Perverse, Hodge and motivic realizations of étale motives

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Abstract. — Let $k = \mathbb{C}$ be the field of complex numbers (one can also choose a field of characteristic zero k with a fixed embedding of fields $\sigma : k \hookrightarrow \mathbb{C}$).

In this article, we construct Hodge realization functors defined on the triangulated categories of étale motives with rational coefficients. Our construction extends, to every smooth quasi-projective k-scheme, the construction done by M. Nori over a field and relies on the original version of the basic lemma proved by A. Beĭlinson. As in the case considered by M. Nori, the realization functor factors through the bounded derived category of a perverse version of the Abelian category of Nori motives.

Contents

1. Introduction

Let $k = \mathbb{C}$ be the field of complex numbers (one can also choose a field of characteristic zero k with a fixed embedding of fields $\sigma : k \hookrightarrow \mathbb{C}$).

1.1. In the present article, we consider the triangulated categories of étale motives $\mathbf{DA}^{\text{ét}}(-,\mathbb{Q})$ over quasi-projective k-schemes. These categories have been introduced by J. Ayoub in [**2, 3**] and are the $\mathbb{Q}$-linear étale counterpart of the stable homotopy category of schemes of F. Morel and V. Voevodsky . The theory developed in [**3**] provides these categories with a six operations formalism. As shown in [**30, 5**], the category $\mathbf{DA}^{\text{ét}}(k,\mathbb{Q})$ is equivalent to the triangulated category of motives $\mathrm{DM}(k,\mathbb{Q})$ considered by V. Voevodsky. Hence the category $\mathrm{DM}_{gm}(k,\mathbb{Q})$ of *geometric motives* of [**43**] can be seen as a full subcategory of $\mathbf{DA}^{\text{ét}}(k,\mathbb{Q})$.

1.2. As part of the vision of A. Grothendieck, these categories should have realization functors. For Betti cohomology (see [**4**]) or ℓ-adic cohomology (see [**20, 19, 21, 22, 5**]) such functors have been constructed.

On the Hodge theoretic side however the picture is far from being complete as the only realization functor available is defined over $\mathrm{Spec}(k)$. Let $\mathrm{MHS}^p_{\mathbb{Q}}$ be the Abelian category of polarizable mixed $\mathbb{Q}$-Hodge structures. Three different construction of such a realization functor

$$\mathrm{DM}_{gm}(k,\mathbb{Q}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{MHS}^p_{\mathbb{Q}})$$

have been given in the literature: one due to M. Levine [**28**], one due to A. Huber [**20, 19**] and one due to M. Nori (though unpublished Nori's construction has been sketched in [**29, 33**]). The first two constructions do not use directly the category of polarizable mixed Hodge structures, they use instead as target the more flexible category of polarizable mixed Hodge complexes $\mathrm{D}^{\mathrm{b}}_{\mathscr{H}^p}$. This category was defined by A. Beĭlinson in [**6**] where he also constructs an equivalence of categories

$$\mathrm{D}^{\mathrm{b}}(\mathrm{MHS}^p_{\mathbb{Q}}) \to \mathrm{D}^{\mathrm{b}}_{\mathscr{H}^p}.$$

However such an equivalence is not available in higher dimension though partial results have been obtained in [**23**]. They are not sufficient to get a realization except perhaps on the triangulated category of smooth motives. Let us also mention that Levine's construction is also indirect as the source category is rather is own category of motives $\mathcal{DM}(k,\mathbb{Q})$ (known to be equivalent to $\mathrm{DM}_{gm}(k,\mathbb{Q})$ by [**28**]).

1.3. The approach we generalize to higher dimension in this work is the construction due to M. Nori. Recall that, using a Tannakian approach, he has defined an Abelian category of mixed motives $\mathrm{NMM}(k)$ over k. Roughly speaking, being a motive in $\mathrm{NMM}(k)$ is the best structure that one can put on the relative homology of a pair of k-varieties. In particular, as the relative homology of pairs carries a (polarizable) mixed Hodge structure, one has a faithful exact functor

$$\mathrm{NMM}(k) \to \mathrm{MHS}^p_{\mathbb{Q}}.$$

Using the the so-called «basic lemma», a special case of a more general result on perverse sheaves due to A. Beĭlinson, M. Nori constructs a finer realization functor

$$\mathrm{DM}_{gm}(k) \to \mathrm{D}^{\mathrm{b}}(\mathrm{NMM}(k)). \tag{1}$$

1.4. In this work we use the original version of the basic lemma proved by A. Beĭlinson to extend the construction of M. Nori to all smooth quasi-projective k-schemes. Recall that if $\mathscr{A}$ is a $\mathbb{Q}$-linear Abelian category, then the Yoneda functor

$$\mathsf{i} : \mathscr{A} \to \mathbf{Sha}(\mathscr{A}, \mathbb{Q}),$$

where $\mathbf{Sha}(\mathscr{A}, \mathbb{Q})$ is the Abelian category of additive sheaves of $\mathbb{Q}$-vector spaces on $\mathscr{A}$ for the topology defined by epimorphisms, is exact and fully faithful (this is the Gabriel-Quillen embedding). Moreover it induces a fully faithful functor

$$\mathrm{D^b}(\mathscr{A}) \to \mathrm{D}(\mathbf{Sha}(\mathscr{A}, \mathbb{Q})).$$

Let us state our main results.

Main results. — *Let X be a smooth quasi-projective k-scheme and $\mathscr{M}(X)$ be either the category $\mathscr{N}(X)$ of perverse Nori motives (see* [**24**]*), the category $\mathscr{H}(X) := \mathrm{MHM}(X, \mathbb{Q})$ of mixed Hodge modules* [**38, 39**]*, or the category $\mathscr{P}(X)$ of perverse sheaves.*

1. *We construct two triangulated functors defining an adjunction*

$$\mathsf{RL}^{\mathscr{M}}_X : \mathbf{DA}^{\text{ét}}(X, \mathbb{Q}) \rightleftarrows \mathrm{D}(\mathbf{Sha}(\mathscr{M}(X)), \mathbb{Q}) : \mathsf{RR}^{\mathscr{M}}_X$$

   the right hand side being the unbounded derived category of $\mathbf{Sha}(\mathscr{M}(X), \mathbb{Q})$.
2. *Let $\mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(X, \mathbb{Q})$ be the full triangulated category of constructible étale motives. The left adjoint $\mathsf{RL}^{\mathscr{M}}_X$ induces then a triangulated functor*

$$\mathsf{RL}^{\mathscr{M}}_X : \mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(X, \mathbb{Q}) \to \mathrm{D^b}(\mathscr{M}(X))$$

   which gives back (1) *when $X = \mathrm{Spec}(\mathbb{C})$.*
3. *If $a : Y \to X$ is smooth quasi-projective morphism of k-schemes and Y is affine, then the image of the homological motive $\mathrm{M}_X(Y)$ under $\mathsf{RL}^{\mathscr{H}}_X$ is isomorphic to the Hodge homology complex $a^{\mathscr{H}}_! a^!_{\mathscr{H}}(\mathbb{Q}^{\mathscr{H}}_X)$ where*

$$a^{\mathscr{H}}_! : \mathrm{D^b}(\mathrm{MHM}(Y, \mathbb{Q})) \rightleftarrows \mathrm{D^b}(\mathrm{MHM}(X, \mathbb{Q})) : a^!_{\mathscr{H}}$$

   are the extraordinary adjoint functors part of the formalism of the six operations developed by M. Saito.

By construction there are $\mathbb{Q}$-linear faithful exact functors $\mathsf{R}^{\mathscr{H}}_X : \mathscr{N}(X) \to \mathrm{MHM}(X, \mathbb{Q})$ and $\mathsf{rat}^{\mathscr{H}}_X : \mathrm{MHM}(X, \mathbb{Q}) \to \mathscr{P}(X)$ (the last one associates the underlying perverse sheaf of a mixed Hodge modules). The functors $\mathsf{RL}^{\mathscr{H}}_X$ and $\mathsf{RL}^{\mathscr{P}}_X$ are obtained from $\mathsf{RL}^{\mathscr{N}}_X$ via these functors.

However, for readers interested in the Hodge realization solely, let us note that the present work is completely independant of [**24**]. The construction does not need the categories of perverse motives of loc.cit. and can be done directly using mixed Hodge modules.

2. Recollection on étale motives

In this section we briefly recall the construction of the categories $\mathbf{DA}^{\text{ét}}(X,\mathbb{Q})$ of étale motives over a quasi-projective k-scheme X and some of their properties. For model categories introduced by D. Quillen in [**36**] we refer e.g. to [**16, 17**].

2.1. The triangulated categories $\mathbf{DA}^{\text{ét}}(X,\mathbb{Q})$ have been introduced in [**2, 3**], where they are particular cases of the categories $\mathbf{SH}_{\mathfrak{M}}(X)$ obtained by choosing the topology to be the étale topology and the model category $\mathfrak{M}$ of coefficients to be the model category $\mathsf{Ch}(\mathbb{Q})$ of chain complexes of $\mathbb{Q}$-vector spaces. They are the $\mathbb{Q}$-linear étale counterpart of the stable homotopy category of X-schemes of F. Morel and V. Voevodsky (see [**25, 32, 42**]) and have been studied in further details in [**5**].

They are part of a stable homotopic 2-functor $\mathbf{DA}^{\text{ét}}(-,\mathbb{Q})$ on the category of quasi-projective k-schemes as defined in [**2**, Définition 2.4.13]. The theory developed by J. Ayoub in [**2, 3**] provides, for these triangulated categories, a six operations formalism as envisioned by A. Grothendieck,

We consider ultimately the full triangulated category $\mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(X,\mathbb{Q})$ of constructible motives, defined as the smallest triangulated subcategory of $\mathbf{DA}^{\text{ét}}(X,\mathbb{Q})$ stable by direct factors and containing the homological motives of smooth quasi-projective X-schemes. As shown in [**2**, Scholie 2.2.34] these categories of constructible motives are stable under the six operations.

2.2. If $\mathscr{A}$ is an additive category, we denote by $\mathsf{Ch}(\mathscr{A})$ the category of cochain complexes of objects in $\mathscr{A}$. Let Λ be a commutative ring. We denote simply by $\mathsf{Ch}(\Lambda) := \mathsf{Ch}(\mathsf{Mod}(\Lambda))$ the category of chain complexes of Λ-modules. We consider on $\mathsf{Ch}(\Lambda)$ the projective model category structure for which the weak equivalences are the quasi-isomorphisms and the fibrations are the epimorphisms.

2.3. Let X be a quasi-projective k-scheme. Let Sm/X be the category of smooth quasi-projective X-schemes. The construction of the category $\mathbf{DA}^{\text{ét}}(X,\mathbb{Q})$ starts with the category $\mathbf{PSh}(\mathsf{Sm}/X,\mathsf{Ch}(\mathbb{Q}))$ of presheaves of $\mathbb{Q}$-vector spaces endowed with its projective model structure: the fibrations (resp. equivalences) are the maps of presheaves of complexes $\mathscr{X}\to\mathscr{Y}$ such that $\mathscr{X}(Y)\to\mathscr{Y}(Y)$ is a fibration (resp. an equivalence) in $\mathsf{Ch}(\mathbb{Q})$ for every $Y\in\mathsf{Sm}/X$.

A left Bousfield localization of this projective model structure provides the ét-local model structure. For the ét-local structure, the weak equivalences are the morphisms of complexes of presheaves that induce isomorphims on the étale sheafification of the homology presheaves. Note that the étale sheafification functor induces then an equivalence of triangulated categories

$$a_{\text{ét}} : \mathsf{Ho}_{\text{ét}}(\mathbf{PSh}(\mathsf{Sm}/X,\mathsf{Ch}(\mathbb{Q}))) \xrightarrow{\sim} \mathrm{D}(\mathbf{Sh}_{\text{ét}}(\mathsf{Sm}/X,\mathbb{Q}))$$

where the left-hand side is the homotopy category for the ét-local projective model structure and the right-hand side is the unbounded derived category of the Abelian category of étale sheaves of $\mathbb{Q}$-vector spaces on Sm/X (see [**3**, Corollaire 4.4.42] for a proof).

The ét-local model structure is then further localized with respect to the class of maps

$$\mathbf{A}^1_Y \otimes \mathbb{Q} \to Y \otimes \mathbb{Q}$$

where $Y \in \mathsf{Sm}/X$. The left Bousfield localization of the ét-local model structure with respects to the above maps is called the $(\mathbf{A}^1, \text{ét})$-local projective model structure. Its homotopy category

$$\mathbf{DA}^{\text{eff},\text{ét}}(X, \mathbb{Q}) := \mathsf{Ho}_{\mathbf{A}^1,\text{ét}}(\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})))$$

is called the category of effective étale motives (with rational coefficients).

The last step of the construction is the stabilization. Let T_X be the presheaf

$$T_X := \frac{\mathbf{G}_{m,X} \otimes \mathbb{Q}}{X \otimes \mathbb{Q}}.$$

Consider the category $\mathsf{Spt}^{\Sigma}_{T_X}(\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})))$ of symmetric T_X-spectrum of presheaves of complexes of $\mathbb{Q}$-vector spaces (see [**3**, Définition 4.3.6]). The $(\mathbf{A}^1, \text{ét})$-local projective model structure induces on it a model structure (see [**3**, Définition 4.3.29]): the so-called $(\mathbf{A}^1, \text{ét})$-local stable projective model structure. Its homotopy category

$$\mathbf{DA}^{\text{ét}}(X, \mathbb{Q}) := \mathsf{Ho}_{(\mathbf{A}^1,\text{ét})-\text{st}}(\mathsf{Spt}^{\Sigma}_{T}(\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))))$$

is the triangulated category of étale motives with rational coefficients.

With a scheme $Y \in \mathsf{Sm}/X$ is a associated a homological motive $\mathrm{M}_X(Y)$ given by the symmetric T_X-spectrum $\mathsf{Sus}^0_{T_X,\Sigma}(X \otimes \mathbb{Q})$.

2.4. It follows from [**11**] that the fibrant objects for the ét-local projective model structure are the fibrant objects for the projective model structure that satisfies étale descent (see e.g. [**11**, Définition 4.3] or [**9**, Définition 3.2.5, §3.2.9] for the definition). Working with rational coefficients simplifies a lot the description these ét-local fibrant objects.

It follows from [**44**, Proposition 3.8] and [**9**, Theorem 3.3.23] that an object $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ is fibrant for the ét-local projective model structure if and only if it is fibrant for the projective model structure, satisfies elementary Galois descent (in the sense of Definition A.2), the B.G. property in the Nisnevich topology and is such that $\mathscr{X}(\emptyset)$ is acyclic.

As a consequence the fibrant object for the $(\mathbf{A}^1$,ét)-local projective model structure are the presheaves $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ that are fibrant for the projective model structure, satisfies Galois descent, the $\mathbf{A}^1$-B.G. property in the Nisnevich topology and such that $\mathscr{X}(\emptyset)$ is acyclic.

By [**31**, Theorem A.14], if an object $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ satisfies the $\mathbf{A}^1$-B.G. property in the Zariski topology and the affine B.G. property in the Nisnevich topology, then it satisfies the B.G. property in the Nisnevich topology.

This description may be reinterprated as follows:

Proposition 2.1. — *The $(\mathbf{A}^1$,ét$)$-local projective model structure on the category $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ is the left Bousfield localization of the projective model structure* *with respect to the following classes of maps:*

$$\begin{tikzcd}
\emptyset\otimes\mathbb{Q} \arrow[d] & (Y'\otimes\mathbb{Q})^G \arrow[d] & \mathbf{A}^1_Y\otimes\mathbb{Q} \arrow[d] & (V\otimes\mathbb{Q}) \arrow[r] & (U\otimes\mathbb{Q})\oplus(E\otimes\mathbb{Q}) \arrow[d] \\
0 & Y\otimes\mathbb{Q} & Y\otimes\mathbb{Q} & & (Y\otimes\mathbb{Q})
\end{tikzcd} \tag{2}$$

where $r : Y' \to Y$ is a Galois cover with Galois group G and

$$\begin{tikzcd}
V \arrow[r, "v"] \arrow[d, "e'"'] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\
U \arrow[r, "u"] & Y
\end{tikzcd}$$

is either an elementary Zariski square of or an elementary affine Nisnevich square.

2.5. The category $\mathbf{DA}^{\text{ét}}(k,\mathbb{Q})$ is equivalent to the triangulated category of motives $\mathrm{DM}(k,\mathbb{Q})$ considered by V. Voevodsky. Indeed, there exist two canonical functors

$$\begin{tikzcd}
\mathbf{DA}^{\text{ét}}(k,\mathbb{Q}) \arrow[r] & \mathrm{DM}^{\text{ét}}(k,\mathbb{Q}) \\
& \mathrm{DM}(k,\mathbb{Q}) \arrow[u]
\end{tikzcd}.$$

In this diagram, the horizontal functor is an equivalence of categories by [**5**, Théorème B.1]. Besides, the vertical one is an equality, since the considered categories have the same objects and arrows (see [**30**, Theorem 14.30, Lemma 14.21]). Hence the category $\mathrm{DM}_{gm}(k,\mathbb{Q})$ of *geometric motives* of [**43**] can be seen as a full subcategory of $\mathbf{DA}^{\text{ét}}(k,\mathbb{Q})$, that contains the additive category $\mathsf{M}_{\mathsf{rat}}(k,\mathbb{Q})$ of Chow motives (over k with rational coefficients).

In [**9**, Définition 14.2.1] D.-C. Cisinski and F. Déglise have introduced the category $\mathrm{DM}_{\mathrm{B}}(X)$ of Beĭlinson motives. As shown in [**9**, Theorem 15.2.16] this category turns out to be equivalent to the previously defined $\mathbf{DA}^{\text{ét}}(X,\mathbb{Q})$. Note that the category of Beĭlinson motives is $\mathbb{Q}$-linear and was defined only after J. Ayoub introduced and constructed the six operations formalism on the category of étale motives.

3. Perverse homology of pairs

Let $k = \mathbb{C}$ be the field of complex numbers (one can also choose a field of characteristic zero k with a fixed embedding of fields $\sigma : k \hookrightarrow \mathbb{C}$).

3.1. Let X be a quasi-projective k-scheme. To keep notations short, we denote by $\mathscr{P}(X)$ the category of perverse sheaves (or the full subcategory $\mathscr{P}(X)^{\mathrm{go}}$ of perverse sheaves of geometric origins [**8**, 6.2.4]) and by $\mathscr{H}(X)$ the category of mixed Hodge modules $\mathrm{MHM}(X,\mathbb{Q})$ introduced by M. Saito in [**38, 39**] (or the full subcategory $\mathrm{MHM}(X,\mathbb{Q})^{\mathrm{go}}$ of mixed Hodge modules of geometric origins [**40**, (2.6) Définition]).

Let $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. Recall that the derived categories $\mathrm{D^b}(\mathscr{M}(X))$, as X runs over quasi-projective k-schemes, are endowed with a six functors formalism

$$\begin{tikzcd} \mathrm{D^b}(\mathscr{M}(X)) \arrow[r, shift right, "f_*^{\mathscr{M}}"'] & \mathrm{D^b}(\mathscr{M}(Y)) \arrow[l, shift right, "f^*_{\mathscr{M}}"'] \arrow[r, shift left, "f_!^{\mathscr{M}}"] & \mathrm{D^b}(\mathscr{M}(X)). \arrow[l, shift left, "f^!_{\mathscr{M}}"] \end{tikzcd}$$

We denote by

$$\mathsf{H}^i_{\mathscr{M}} : \mathrm{D^b}(\mathscr{M}(X)) \to \mathscr{M}(X) \quad i \in \mathbb{Z}$$

the cohomological functor associated with the usual t-structure. We set $\mathsf{H}_i^{\mathscr{M}} = \mathsf{H}^{-i}_{\mathscr{M}}$. In this section we fix an integer $d \in \mathbb{N}$ (later taken to be the dimension of X).

3.2. A relative X-triplet is a triplet (Y, Z, i) where Y is quasi-projective X-scheme, Z is a closed subset of Y and $i \in \mathbb{Z}$ is an integer.

Definition 3.1. — Let $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$ and (Y, Z, i) be a relative X-triplet. We set

$$\mathsf{TH}_X^{\mathscr{M}}(Y, Z, i) := \mathsf{H}^{2d-i}_{\mathscr{M}}(a_!^{\mathscr{M}}(u_*^{\mathscr{M}} u^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})))$$

where $u : U \hookrightarrow Y$ is the open immersion of the complement of Z in Y and $a : Y \to X$ is the structural morphism.

Note that by definition $\mathsf{TH}_X^{\mathscr{M}}(Y, Z, i)$ is an object in $\mathscr{M}(X)$ which only depends on the reduced structure of Y . Recall that $\mathbb{Q}_Y^{\mathscr{M}}$ is not in general an object in $\mathscr{M}(Y)$. If Y is smooth over k of pure dimension n, then $\mathbb{Q}_Y^{\mathscr{M}}[n]$ belongs to $\mathscr{M}(Y)$.

Remark 3.2. — With the notations of [**24**], one has

$$\mathsf{TH}_X^{\mathscr{M}}(Y, Z, i) = \mathsf{T}_X^{\mathscr{M}}(Y, Z, i - 2d).$$

3.3. Let (Y_1, Z_1, i) and (Y_2, Z_2, i) be relative X-triplets. Assume that $f : Y_2 \to Y_1$ is a morphism of X-schemes, such that $f(Z_2) \subseteq Z_1$. Then there are morphisms in $\mathscr{M}(X)$

$$f_\star^{\mathscr{M}} : \mathsf{TH}_X^{\mathscr{M}}(Y_2, Z_2, i) \to \mathsf{TH}_X^{\mathscr{M}}(Y_1, Z_1, i) \tag{3}$$

such that if (Y_3, Z_3, i) is a relative X-triplet, and $g : Y_3 \to Y_2$ is a morphism of X-schemes such that $g(Z_3) \subseteq Z_2$, then

$$f_\star^{\mathscr{M}} \circ g_\star^{\mathscr{M}} = (fg)_\star^{\mathscr{M}}.$$

Recall that the morphism (3) is obtained as follows. Consider the commutative diagram

$$\begin{tikzcd} f^{-1}(U_1) \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & U_2 \arrow[r, "u_2"] & Y_2 \arrow[d, "f"] \arrow[dr, "a_2"] & \\ U_1 \arrow[rr, "u_1"] & & Y_1 \arrow[r, "a_1"'] & X \end{tikzcd}$$

in which U_1 (resp. U_2) is the open complement of Z_1 (resp. Z_2) and all arrows are the canonical morphisms. Using Smooth Base Change and adjunction, we have a morphism in $\mathrm{D^b}(\mathscr{M}(Y_1))$

$$\begin{tikzcd} f_!^{\mathscr{M}}(u_2)_*^{\mathscr{M}}(u_2)^!_{\mathscr{M}}(a_2)^!_{\mathscr{M}} \arrow[r] & f_!^{\mathscr{M}}(u_2)_*^{\mathscr{M}}u_*^{\mathscr{M}}u^!_{\mathscr{M}}(u_2)^!_{\mathscr{M}}(a_2)^!_{\mathscr{M}} \arrow[d, equal] & \\ & f_!^{\mathscr{M}}(u_2)_*^{\mathscr{M}}u_*^{\mathscr{M}}f^!_{\mathscr{M}}(u_1)^!_{\mathscr{M}}(a_1)^!_{\mathscr{M}} \arrow[d, equal] & \\ & f_!^{\mathscr{M}}f^!_{\mathscr{M}}(u_1)_*^{\mathscr{M}}(u_1)^!_{\mathscr{M}}(a_1)^!_{\mathscr{M}} \arrow[r] & (u_1)_*^{\mathscr{M}}(u_1)^!_{\mathscr{M}}(a_1)^!_{\mathscr{M}}. \end{tikzcd}$$

Applying successively $(a_1)_!^{\mathscr{M}}$ and the cohomological functor $\mathsf{H}^{2d-i}_{\mathscr{M}}$ to this morphism, we obtain the morphism (3) in $\mathscr{M}(X)$.

3.4. Now let (Y,Z,i) be a relative X-triplet, and $W \subseteq Z$ be a closed subset. Then we have a boundary morphism

$$\mathsf{TH}^{\mathscr{M}}_X(Y,Z,i) \to \mathsf{TH}^{\mathscr{M}}_X(Z,W,i-1) \tag{4}$$

defined as follows. Consider the commutative diagram

$$\begin{tikzcd} U := Y\setminus Z \arrow[r, "j"] \arrow[rr, bend left, "u"] & Y\setminus W \arrow[r, "v_Y"] & Y \arrow[r, "a"] & X \\ & V := Z\setminus W \arrow[u, "z_V"] \arrow[r, "v"] \arrow[ur, phantom, "\square"] & Z \arrow[u, "z"'] \arrow[ur, "b"'] & \end{tikzcd}$$

where v, v_Y, j are the open immersions, z the closed immersion and a, b the structural morphisms. The localization triangle in $\mathrm{D^b}(\mathscr{M}(Y\setminus W))$

$$(z_V)_!^{\mathscr{M}}(z_V)^!_{\mathscr{M}} \to \mathrm{id} \to j_*^{\mathscr{M}} j^*_{\mathscr{M}} \xrightarrow{+1},$$

applied to $(v_Y)^!_{\mathscr{M}}a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)$, provides a morphism

$$j_*^{\mathscr{M}}u^!_{\mathscr{M}}a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X) \to (z_V)_!^{\mathscr{M}}v^!_{\mathscr{M}}b^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)[1].$$

As z and z_V are closed immersions, applying $(v_Y)_*^{\mathscr{M}}$, yields a morphism

$$u_*^{\mathscr{M}}u^!_{\mathscr{M}}a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X) \to z_!^{\mathscr{M}}v_*v^!_{\mathscr{M}}b^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)[1]$$

By applying $a_!^{\mathscr{M}}$ and the cohomological functor H^{2d-i} one gets the boundary map (4).

3.5. Recall that in [**24**] we have constructed a $\mathbb{Q}$-linear Abelian category $\mathscr{N}(X)$ with a faithful exact functor $\mathscr{N}(X) \to \mathscr{P}(X)$ that factorizes through $\mathrm{MHM}(X,\mathbb{Q})$. By construction, with every relative X-triplet (Y,Z,i) is attached an object $\mathsf{TH}^{\mathscr{N}}_X(Y,Z,i)$ in $\mathscr{N}(X)$. These objects enjoy the same functorialities as previously described. More precisely if (Y_1,Z_1,i) and (Y_2,Z_2,i) are relative X-triplets and $f : Y_2 \to Y_1$ is a morphism of X-schemes, such that $f(Z_2) \subseteq Z_1$, then the category $\mathscr{N}(X)$ contains a morphism

$$f_\star^{\mathscr{N}} : \mathsf{TH}_X^{\mathscr{N}}(Y_2, Z_2, i) \to \mathsf{TH}_X^{\mathscr{N}}(Y_1, Z_1, i) \tag{5}$$

which maps to (3) via the functor $\mathscr{N}(X) \to \mathscr{M}(X)$. Similarly if (Y, Z, i) be a relative X-triplet, and $W \subseteq Z$ be a closed subset, then $\mathscr{N}(X)$ contains a morphism. Then we have a boundary morphism

$$\mathsf{TH}_X^{\mathscr{N}}(Y, Z, i) \to \mathsf{TH}_X^{\mathscr{N}}(Z, W, i-1) \tag{6}$$

compatible again with (4).

3.6. The next lemma is elementary but useful in the sequel.

Lemma 3.3. — *Let (Y, Z, i) be a relative triplet. Then*

$$\cdots \to \mathsf{TH}_X^{\mathscr{M}}(Z, \emptyset, i) \to \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, i) \to \mathsf{TH}_X^{\mathscr{M}}(Y, Z, i) \to \mathsf{TH}_X^{\mathscr{M}}(Z, \emptyset, i-1) \to \cdots \tag{7}$$

is a long exact sequence in $\mathscr{M}(X)$.

Proof. — Since the functor $\mathscr{N}(X) \to \mathscr{P}(X)$ is exact and faithful, we may assume that $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. Apply then the distinguished triangle $z_!^{\mathscr{M}} z^!_{\mathscr{M}} \to \mathrm{Id} \to u_*^{\mathscr{M}} u^*_{\mathscr{M}} \xrightarrow{+1}$ to $a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})$ and take its image by $a_!^{\mathscr{M}}$ to get the distinguished triangle

$$(a \circ z)_!^{\mathscr{M}} (a \circ z)^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}) \to a_!^{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}) \to a_!^{\mathscr{M}} u_*^{\mathscr{M}} u^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}) \xrightarrow{+1} .$$

The associated long exact sequence yields the desired long exact sequence. □

The morphisms (3) and (4) (or (5) and (6) as well) are compatible. More precisely we have the following lemma:

Lemma 3.4. — *Let $f : Y_2 \to Y_1$ be a X-morphism of quasi-projective k-varieties. Let $W_2 \subseteq Z_2$ and $W_1 \subseteq Z_1$ such that $f(Z_2) \subseteq Z_1$ and $f(W_2) \subseteq W_1$. Then the square of morphisms in $\mathscr{M}(X)$*

$$\begin{tikzcd}
\mathsf{TH}_X^{\mathscr{M}}(Y_2, Z_2, i) \arrow[r, "\partial"] \arrow[d, "f_\star^{\mathscr{M}}"] & \mathsf{TH}_X^{\mathscr{M}}(Z_2, W_2, i-1) \arrow[d, "f_\star^{\mathscr{M}}"] \\
\mathsf{TH}_X^{\mathscr{M}}(Y_2, Z_2, i) \arrow[r, "\partial"] & \mathsf{TH}_X^{\mathscr{M}}(Z_2, W_2, i-1)
\end{tikzcd}$$

is commutative.

3.7. We now give some properties of relative $\mathscr{M}$-homology objects needed in the sequel to construct the realization functors.

Lemma 3.5. — *Let $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. Let (Y, Z, i) be a relative X-triplet and*

$$\begin{tikzcd}
V \arrow[r, "v"] \arrow[d, "e'"] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\
U \arrow[r, "u"] & Y
\end{tikzcd}$$

be a Nisnevich square. Then there is a long exact sequence in $\mathscr{M}(X)$:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathsf{TH}^{\mathscr{M}}_X(Y,Z,i+1) \arrow[r] & \mathsf{TH}^{\mathscr{M}}_X(V,V_Z,i) \arrow[d] & & \\
& & \mathsf{TH}^{\mathscr{M}}_X(U,U_Z,i) \oplus \mathsf{TH}^{\mathscr{M}}_X(E,Z_E,i) \arrow[r] & \mathsf{TH}^{\mathscr{M}}_X(Y,Z,i) \arrow[r] & \cdots
\end{tikzcd} \tag{8}$$

where $Z_V := Z \times_X V$, $Z_U := Z \times_X U$ and $Z_E := Z \times_X E$.

Proof. — Let $w : W \hookrightarrow Y$ be an open immersion of the complement of Z in Y. Consider the diagram obtained by base change:

$$\begin{tikzcd}
V_W \arrow[r, "v_W"] \arrow[d, "e'_W"'] \arrow[drr, "w_V"] & E_W \arrow[d, dotted, "e_W"] \arrow[drr, "w_E"] & & \\
& & V \arrow[r, "v"] \arrow[d, "e'"'] & E \arrow[d, "e"] \\
U_W \arrow[r, dotted, "u_W"] \arrow[drr, "w_U"'] & W \arrow[drr, dotted, "w"] & & \\
& & U \arrow[r, "u"] & Y
\end{tikzcd}$$

Let $h = e \circ v = u \circ e'$ and $h_W = e_W \circ v_W = u_W \circ e'_W$. We have a distinguished triangle

$$h^{\mathscr{M}}_! h^!_{\mathscr{M}} \to u^{\mathscr{M}}_! u^!_{\mathscr{M}} \oplus e^{\mathscr{M}}_! e^!_{\mathscr{M}} \to \mathrm{Id} \xrightarrow{+1} .$$

Applying this triangle to $w^{\mathscr{M}}_* w^*_{\mathscr{M}}$, yields the distinguished triangle

$$\begin{tikzcd}
h^{\mathscr{M}}_!(w_V)^{\mathscr{M}}_*(w_V)^*_{\mathscr{M}} h^!_{\mathscr{M}} \arrow[r] & u^{\mathscr{M}}_!(w_U)^{\mathscr{M}}_*(w_U)^*_{\mathscr{M}} u^!_{\mathscr{M}} \oplus e^{\mathscr{M}}_!(w_E)^{\mathscr{M}}_*(w_E)^*_{\mathscr{M}} e^!_{\mathscr{M}} \arrow[d] \\
& w^{\mathscr{M}}_* w^*_{\mathscr{M}} \xrightarrow{+1}
\end{tikzcd} \tag{9}$$

since using Smooth Base Change, we get

$$e^{\mathscr{M}}_! e^!_{\mathscr{M}} w^{\mathscr{M}}_* w^*_{\mathscr{M}} = e^{\mathscr{M}}_!(w_E)^{\mathscr{M}}_*(w_E)^*_{\mathscr{M}} e^!_{\mathscr{M}} \qquad u^{\mathscr{M}}_! u^!_{\mathscr{M}} w^{\mathscr{M}}_* w^*_{\mathscr{M}} = u^{\mathscr{M}}_!(w_U)^{\mathscr{M}}_*(w_U)^*_{\mathscr{M}} u^!_{\mathscr{M}}$$

$$h^{\mathscr{M}}_! h^!_{\mathscr{M}} w^{\mathscr{M}}_* w^*_{\mathscr{M}} = h^{\mathscr{M}}_!(w_V)^{\mathscr{M}}_*(w_V)^*_{\mathscr{M}} h^!_{\mathscr{M}}.$$

Applying the triangle (9) to $a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)$ and taking the image under $a^{\mathscr{M}}_!$ yields a new distingushed triangle. The long exact sequence (8) is then the long exact sequence associated with this triangle. □

Corollary 3.6. — *Let (Y, Z, i) be a relative X-triplet and*

$$\begin{tikzcd}
V \arrow[r, "v"] \arrow[d, "e'"'] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\
U \arrow[r, "u"] & Y
\end{tikzcd}$$

be a Nisnevich square. Then there is a short exact sequence in $\mathscr{N}(X)$:

$$\mathsf{TH}^{\mathscr{N}}_X(V,V_Z,i) \to \mathsf{TH}^{\mathscr{N}}_X(U,U_Z,i) \oplus \mathsf{TH}^{\mathscr{N}}_X(E,Z_E,i) \to \mathsf{TH}^{\mathscr{N}}_X(Y,Z,i) \tag{10}$$

where $Z_V := Z \times_X V$, $Z_U := Z \times_X U$ and $Z_E := Z \times_X E$.

Proof. — This follows from Lemma 3.5 since the functor $\mathscr{N}(X) \to \mathscr{P}(X)$ is exact and faithful. (Note that this is not clear a priori that the boundary morphism in the long exact sequence (8) exists in the category of perverse Nori motives $\mathscr{N}(X)$). $\square$

Lemma 3.7. — *Let (Y, Z, i) be a relative triplet and $p : Y' \to Y$ be a Galois covering with Galois group G. Then the morphism*

$$\mathsf{TH}^{\mathscr{M}}_X(Y', Z', i)^G \to \mathsf{TH}^{\mathscr{M}}_X(Y, Z, i)$$

is an isomorphism.

Proof. — Since the functor $\mathscr{N}(X) \to \mathscr{P}(X)$ is exact and faithful, we may assume that $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. Let $z : Z \hookrightarrow Y$ be a closed immersion and $u : U \hookrightarrow Y$ be the open immersion of the complement. Consider their pullbacks $u' : U' \hookrightarrow Y'$ and $z' : Z' \hookrightarrow Y'$ along p. Let $A \in \mathrm{D}^{\mathrm{b}}(\mathscr{M}(Y))$. Then the have the commutative diagram

$$\begin{tikzcd}
\left[p^{!}_{\mathscr{M}}(z')^{\mathscr{M}}_{!}(z')^{!}_{\mathscr{M}}p^{!}_{\mathscr{M}}(A)\right]^G \arrow[r] \arrow[d] & \left[p^{\mathscr{M}}_{!}p^{!}_{\mathscr{M}}(A)\right]^G \arrow[r] \arrow[d] & \left[p^{\mathscr{M}}_{!}(u')^{\mathscr{M}}_{*}(u')^{*}_{\mathscr{M}}p^{!}_{\mathscr{M}}(A)\right]^G \arrow[r, "+1"] \arrow[d] & {} \\
z^{\mathscr{M}}_{!}z^{!}_{\mathscr{M}}(A) \arrow[r] & A \arrow[r] & u^{\mathscr{M}}_{*}u^{!}_{\mathscr{M}}(A) \arrow[r, "+1"] & {}
\end{tikzcd}$$

where lines are distinguished triangles in $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(Y))$. The first two vertical arrows are isomorphism by étale descent for Betti cohomology, hence so is the map

$$\left[p^{\mathscr{M}}_{!}(u')^{\mathscr{M}}_{*}(u')^{*}_{\mathscr{M}}p^{!}_{\mathscr{M}}(A)\right]^G \to u^{\mathscr{M}}_{*}u^{!}_{\mathscr{M}}(A).$$

This implies that the maps $\mathsf{TH}^{\mathscr{M}}_X(Y', Z', i)^G \to \mathsf{TH}^{\mathscr{M}}_X(Y, Z, i)$ are isomorphisms for all integer $i \in \mathbb{Z}$. $\square$

Lemma 3.8. — *Let Y be a quasi-projective k-scheme and $T \to Y$ be a finite rank vector bundle. Then for every integer $i \in \mathbb{Z}$*

$$\mathsf{TH}^{\mathscr{M}}_X(T, Y, i) = 0$$

where Y is embedded in T via the zero section.

Proof. — Since the functor $\mathscr{N}(X) \to \mathscr{P}(X)$ is exact and faithful, we may assume that $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. Now consider the zero section $\sigma : Y \to T$ and denote the open immersion of the complement by u. Let $p : T \to Y$ be the projection. By homotopy invariance

$$p^{\mathscr{M}}_{!}p^{!}_{\mathscr{M}} \to \mathrm{Id}$$

is an isomorphism. We have the distinguished triangle $\sigma^{\mathscr{M}}_{!}\sigma^{!}_{\mathscr{M}} \to \mathrm{Id} \to u^{\mathscr{M}}_{*}u^{!}_{\mathscr{M}} \xrightarrow{+1}$. But $p \circ \sigma = \mathrm{Id}$, hence the canonical morphism

$$a^{\mathscr{M}}_{!}p^{\mathscr{M}}_{!}\sigma^{\mathscr{M}}_{!}\sigma^{!}_{\mathscr{M}}p^{!}_{\mathscr{M}}a^{!}_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X) \to a^{\mathscr{M}}_{!}p^{\mathscr{M}}_{!}p^{!}_{\mathscr{M}}a^{!}_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)$$

is an isomorphism, and thus

$$a^{\mathscr{M}}_{!}p^{\mathscr{M}}_{!}u^{\mathscr{M}}_{*}u^{!}_{\mathscr{M}}p^{!}_{\mathscr{M}}a^{!}_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X) = 0$$

in $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$. In particular, for all integer $i \in \mathbb{Z}$, we have the vanishing $\mathsf{TH}^{\mathscr{M}}_X(T, Y, i) = 0$. $\square$

Lemma 3.9. — *Let (Y,Z,i) be a relative X-triplet. We have a decomposition into direct summands*

$$\mathsf{TH}^{\mathscr{M}}_X(\mathbf{G}_{m,Y},\mathbf{G}_{m,Z},i) = \mathsf{TH}^{\mathscr{M}}_X(Y,Z,i) \oplus \mathsf{TH}^{\mathscr{M}}_X(Y,Z,i-1)(1). \tag{11}$$

If $W \subseteq Z$ be a closed subset, then the decomposition (11) *is compatible with boundary maps i.e. the square*

$$\begin{tikzcd}
\mathsf{TH}^{\mathscr{M}}_X(\mathbf{G}_{m,Y},\mathbf{G}_{m,Z},i) \arrow[r] \arrow[d, equal] & \mathsf{TH}^{\mathscr{M}}_X(\mathbf{G}_{m,Z},\mathbf{G}_{m,W},i-1) \arrow[d, equal] \\
\mathsf{TH}^{\mathscr{M}}_X(Y,Z,i) \oplus \mathsf{TH}^{\mathscr{M}}_X(Y,Z,i-1)(1) \arrow[r] & \mathsf{TH}^{\mathscr{M}}_X(Z,W,i-1) \oplus \mathsf{TH}^{\mathscr{M}}_X(Z,W,i-2)(1)
\end{tikzcd}$$

is commutative.

Proof. — Again we may assume $\mathscr{N} \in \{\mathscr{H},\mathscr{P}\}$. Let $z : Z \hookrightarrow Y$ be a closed immersion, and $u : U \hookrightarrow Y$ be its open complement. We denote by $\pi : \mathbf{G}_{m,k} \to \mathrm{Spec}(k)$ the projection. Recall that there is an isomorphism

$$\pi_!^{\mathscr{M}}\pi^!_{\mathscr{M}}(\mathbb{Q}_k^{\mathscr{M}}) = \mathbb{Q}_k^{\mathscr{M}} \oplus \mathbb{Q}_k^{\mathscr{M}}(1)[1].$$

in $\mathrm{D^b}(\mathscr{M}(\mathrm{Spec}(k)))$. We have an isomorphism

$$(u\times_k \mathrm{Id})^!_{\mathscr{M}}(a\times_k\pi)^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}) = u^!_{\mathscr{M}}a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}) \boxtimes \pi^!_{\mathscr{M}}(\mathbb{Q}_k^{\mathscr{M}}).$$

The object $(a\times_k\pi)_!^{\mathscr{M}}(u\times_k\mathrm{Id})_*^{\mathscr{M}}(u\times_k\mathrm{Id})^!_{\mathscr{M}}(a\times_k\pi)^!(\mathbb{Q}_X^{\mathscr{M}})$ of $\mathrm{D^b}(\mathscr{M}(X))$ is therefore isomorphic to

$$\begin{aligned}(a_!^{\mathscr{M}}u_*^{\mathscr{M}}u^!_{\mathscr{M}}a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}))\boxtimes(\pi_!^{\mathscr{M}}\pi^!_{\mathscr{M}}(\mathbb{Q}_k^{\mathscr{M}}) = {} & a_!^{\mathscr{M}}u_*^{\mathscr{M}}u^!_{\mathscr{M}}a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}) \\ & \oplus (a_!^{\mathscr{M}}u_*^{\mathscr{M}}u^!_{\mathscr{M}}a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}))(1)[1].\end{aligned}$$

This yields the decomposition into direct summands in (11). The commutativity of the square is easy to verify from the definition of boundary maps. □

4. Perverse cellular complexes

We assume that X is a *smooth* quasi-projective k-variety. We may assume that X is connected of dimension d. We denote by SmAff/X the category of smooth quasi-projective X-schemes that are affine.

In this section, given a scheme $Y \in \mathsf{SmAff}/X$, we use the basic Lemma [**7**, Lemma 3.3] proved by A. Beĭlinson to associate with certain stratifications of Y an explicit complex of mixed Hodges modules, perverse sheaves or perverse motives that computes its relative homology. This construction is the crucial step towards the realization functor.

4.1. Let Y be a quasi-projective k-scheme. A stratification $Y_\bullet$ of Y is an sequence of closed subsets of Y

$$Y_\bullet : \cdots \subseteq Y_i \subseteq Y_{i+1} \subseteq \cdots \qquad i \in \mathbb{Z}$$

such that $\dim(Y_i) \leqslant i$ for every integer $i \in \mathbb{Z}$, and such that $Y_n = Y$ for some integer n. Note that the condition on dimensions implies that $Y_{-1} = \emptyset$.

Let $Y_\bullet$ and $Y'_\bullet$ be two stratifications of Y. We say that $Y'_\bullet$ is finer than $Y_\bullet$, and write $Y_\bullet \leqslant Y'_\bullet$, if $Y_i \subseteq Y'_i$ for every integer $i \in \mathbb{Z}$. This defines an order relation on the set $\mathscr{S}_Y$ of all stratifications of Y. The ordered set $\mathscr{S}_Y$ if filtered. Indeed, since

$$\dim(Y_i \cup Y'_i) \leqslant i,$$

there is a stratification $Y''_\bullet$ given by $Y''_i := Y_i \cup Y'_i$ and it is finer than $Y_\bullet$ and $Y'_\bullet$.

Let $f : Y \to Y'$ be a morphism of schemes of quasi-projective X-schemes and $Y_\bullet$ be a stratification of Y. Let

$$Y'_i := \overline{f(Y_i)}$$

be the closure of the image of Y_i in Y'. Then $Y'_\bullet$ is a stratification of Y'. Indeed by [**14**, Théorème (4.1.2)], for every integer $i \in \mathbb{Z}$,

$$\dim(Y'_i) \leqslant \dim(Y_i) \leqslant i.$$

We call this stratification the image of $Y_\bullet$ by f and write (abusively $f(Y_\bullet) := Y'_\bullet$).

This defines a functor $f_\sharp : \mathscr{S}_Y \to \mathscr{S}_{Y'}$. Let $f' : Y' \to Y''$ be another morphism of quasi-projective X-schemes. Then for every integer $i \in \mathbb{Z}$

$$\overline{f'\left(\overline{f(Y_i)}\right)} = \overline{f'\left(f(Y_i)\right)}.$$

This means that the two stratifications $f'_\sharp(f_\sharp(Y_\bullet))$ and $(f'f)_\sharp(Y_\bullet)$ are the same. In other words $f'_\sharp \circ f_\sharp = (f' \circ f)_\sharp$ as functors.

4.2. The following definition is essential in the sequel:

Definition 4.1. — Let Y be a quasiprojective X-scheme. A stratification

$$Y_\bullet : \emptyset = Y_{-1} \subseteq Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_{n-1} \subseteq Y_n = Y$$

of Y is said to be cellular if and only if for every $i \in \mathbb{Z}$ the following conditions are satisfied:

- if $\dim(Y_i) = i$, then $\dim(Y_{i-1}) \leqslant i-1$ and for every $k \in \mathbb{Z}$, $k \neq i$, one has
$$\mathsf{TH}^{\mathscr{M}}_X(Y_i, Y_{i-1}, k) = 0$$
in $\mathscr{M}(X)$;
- if $\dim(Y_i) \leqslant i-1$, then $Y_i = Y_{i-1}$.

Note that in the second case $\mathsf{TH}^{\mathscr{M}}_X(Y_i, Y_{i-1}, k) = 0$ for every $k \in \mathbb{Z}$. Assume $Y \neq \emptyset$ and let $n \in \mathbb{N}$ be the smallest integer such that $Y_n = Y$. Then we must have $n \leqslant \dim(Y)$ by the second condition. For a stratification $Y_\bullet$ to be cellular is not a property with respect to Y but with respect to the morphism $Y \to X$. This will cause no confusion in the sequel as our scheme X is fixed once and for all.

If $f : Y \to Y'$ is a morphism of quasi-projective X-schemes, the image of a cellular stratification under the functor $f_\sharp$ may not be a cellular stratification. So as far as functoriality is concerned it is better to consider all stratifications and not only the cellular ones.

Remark 4.2. — The long exact sequence (7) provides the short exact sequences

$$\mathsf{TH}^{\mathscr{M}}_X(Y_i, Y_{i-1}, k+1) \to \mathsf{TH}^{\mathscr{M}}_X(Y_{i-1}, \emptyset, k) \to \mathsf{TH}^{\mathscr{M}}_X(Y_i, \emptyset, k) \to \mathsf{TH}^{\mathscr{M}}_X(Y_i, Y_{i-1}, k).$$

In particular, if $Y_\bullet$ is a cellular stratification of Y, then for $k < i-1$ or $k > i$ the canonical morphism

$$\mathsf{TH}^{\mathscr{M}}_X(Y_{i-1}, \emptyset, k) \to \mathsf{TH}^{\mathscr{M}}_X(Y_i, \emptyset, k)$$

is an isomorphism in $\mathscr{M}(X)$. This implies that, for $k < i$ or $k > n$, the morphism

$$\mathsf{TH}^{\mathscr{M}}_X(Y_i, \emptyset, k) \to \mathsf{TH}^{\mathscr{M}}_X(Y, \emptyset, k)$$

is an isomorphism in $\mathscr{M}(X)$.

Remark 4.3. — Let $Y_\bullet$ be a cellular stratification of Y and n be an integer such that $Y = Y_n$. It is easy to see by induction that $\mathsf{TH}^{\mathscr{M}}_X(Y, \emptyset, i) = 0$ for every integer $i \in \mathbb{Z}$ such that $i < 0$ or $i > n$. Indeed for $n = 0$ this follows from Definition 4.1. For n bigger, Remark 4.2 implies that

$$\mathsf{TH}^{\mathscr{M}}_X(Y_{n-1}, \emptyset, i) \xrightarrow{\simeq} \mathsf{TH}^{\mathscr{M}}_X(Y_n, \emptyset, i) = \mathsf{TH}^{\mathscr{M}}_X(Y, \emptyset, i)$$

for $i < 0$ or $i > n$ and vanishing follows by induction.

The following result is an immediate application of [**7**, Lemma 3.3]:

Lemma 4.4. — *Let $a : Y \to X$ be an affine morphism of finite type. Assume that* $\dim(Y) = n$. *For a closed subset W such that* $\dim(W) \leqslant n-1$, *there exist a closed subset Z of Y containing W and such that* $\dim(Z) \leqslant n-1$ *and for every integer $i \neq n$*

$$\mathsf{TH}^{\mathscr{M}}_X(Y, Z, i) = 0.$$

If Y is integral then we may choose Z such that its reduced open complement is smooth over k.

Note that we do not have to assume here that the scheme Y is affine, but solely that the morphism $a : Y \to X$ is affine.

Proof. — As the functor $\mathscr{N}(X) \to \mathscr{P}(X)$ is exact and faithful, we may assume that $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. We may assume that Y is reduced. By replacing W by the union of W and the irreductible components of Y of dimension $\leqslant n-1$, we may assume that W contains all the irreductible components of Y of dimension $\leqslant 1$. Then $Y \setminus W$ is open in Y and of pure dimension n (i.e. all its irreductible components are of dimension n). As k is of characteristic zero, by [**15**, Proposition (17.15.12)], there is an affine dense open subset V in $Y \setminus W$ which is smooth over k. Since V is smooth of pure dimension n, $\mathbb{Q}^{\mathscr{M}}_V[n]$ is an object in $\mathscr{M}(V)$ and

$$A := v^{\mathscr{M}}_* v^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)[2d-n]$$

belongs to $\mathscr{M}(Y)$. Apply [**7**, Lemma 3.3] to this object $A \in \mathscr{M}(Y)$. This yields an affine open U' in Y such that $\dim(Y \setminus U') \leqslant n-1$ and such that

– the canonical morphism
$$A \to (u')_*^{\mathscr{M}}(u')^*_{\mathscr{M}}A$$
is injective;
– for every $i \in \mathbb{Z} \setminus \{0\}$, one has
$${}^pH^i a_!^{\mathscr{M}}\left((u')_*^{\mathscr{M}}(u')^*_{\mathscr{M}}A\right) = 0$$
in $\mathscr{M}(X)$.

Let U be the intersection of the two dense open subset U' and V and Z its complement in Y. We have $W \subseteq Z$ and $\dim(Z) \leqslant n-1$ and the square of open immersions

$$\begin{tikzcd} U \arrow[r, "j'"] \arrow[d, "j"'] \arrow[dr, phantom, "\square"] & U' \arrow[d, "u'"] \\ V \arrow[r, "v"] & Y \end{tikzcd}$$

is cartesian. By Smooth Base Change
$$\begin{aligned}(u')_*^{\mathscr{M}}(u')^*_{\mathscr{M}}A &= (u')_*^{\mathscr{M}}(u')^*_{\mathscr{M}}v_*^{\mathscr{M}}v^!_{\mathscr{M}}a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})[2d-n] \\ &= (u')_*^{\mathscr{M}}(j')_*^{\mathscr{M}}j^*_{\mathscr{M}}v^!_{\mathscr{M}}a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})[2d-n] \\ &= (u)_*^{\mathscr{M}}(u)^!_{\mathscr{M}}a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})[2d-n]\end{aligned}$$
Hence $\mathsf{TH}_X^{\mathscr{M}}(Y,Z,i) = 0$ for $i \neq n$ and the proof is done. □

Note that for $X = \mathrm{Spec}(k)$ the above lemma is nothing but the so-called Basic Lemma of M. Nori [**34**, Basic Lemma – first form]. As a consquence, the subset $\mathscr{A}_Y$ of cellular stratifications is cofinal in $\mathscr{S}_Y$. Namely we have the following lemma:

Lemma 4.5. — *Let $a : Y \to X$ be an affine morphism of finite type and $Y_\bullet$ be a stratification of Y. Then there exists a cellular stratification of Y finer than $Y_\bullet$.*

Proof. — We construct the stratification by induction. Let us set $Y'_n = Y_n$. Assume that we have constructed $Y'_r \subseteq Y'_{r+1} \subseteq \cdots \subseteq Y'_n = Y$ such that $Y_i \subseteq Y'_i$ and $\dim(Y'_i) \leqslant i$ for every $r \leqslant i \leqslant n$ and such that one of the following condition is satisfied

– $\dim(Y'_i) = i$, $\dim(Y'_{i-1}) \leqslant i-1$ and for every $k \in \mathbb{Z}$, $k \neq i$, one has
$$\mathsf{TH}_X^{\mathscr{M}}(Y'_i, Y'_{i-1}, k) = 0$$
in $\mathscr{M}(X)$;
– $Y'_i = Y'_{i-1}$ and $\dim(Y'_i) \leqslant i-1$.

for $r+1 \leqslant i \leqslant n$. If $\dim(Y'_r) \leqslant r-1$, then we set $Y'_{r-1} = Y'_r$. Otherwise $\dim(Y'_r) = r$ and since $Y_{r-1} \subseteq Y'_r$ and $\dim(Y_{r-1}) \leqslant r-1$, we may apply Lemma 4.4, to obtain a closed subset Y'_{r-1}, such that $Y_{r-1} \subseteq Y'_{r-1} \subseteq Y'_r$, $\dim(Y'_{r-1}) \leqslant r-1$ and
$$\mathsf{TH}_X^{\mathscr{M}}(Y'_r, Y'_{r-1}, i)$$
for every integer $i \neq r$. □

Corollary 4.6. — *Let $a : Y \to X$ be an affine morphism of finite type. There exists a cellular stratification $Y_\bullet$ on Y.*

Proof. — Let n be the dimension of Y. It suffices to apply Lemma 4.5 to the stratification $Y_\bullet$ such that $Y_i = \emptyset$ for $i < n$ and $Y_i = Y$ for $i \geqslant n$. □

Corollary 4.7. — *The ordered subset $\mathscr{A}_Y$ of $\mathscr{S}_Y$ formed by the cellular stratifications is filtered.*

Proof. — Since $\mathscr{S}_Y$ is filtered, this follows immediately from Lemma 4.5. □

4.3. The starting point of main construction are the following complexes associated with stratifications of (affine) quasi-projective X-schemes:

Definition 4.8. — Let Y be a quasi-projective X-scheme. Let $Y_\bullet$ be a stratification of Y. We denote by $\mathsf{TH}^{\mathscr{M}}_X(Y, Y_\bullet)$ the complex in $\mathsf{Ch}(\mathscr{M}(X))$ given by:

$$\cdots \to \mathsf{TH}^{\mathscr{M}}_X(Y_i, Y_{i-1}, i) \to \mathsf{TH}^{\mathscr{M}}_X(Y_{i-1}, Y_{i-2}, i-1) \to \cdots \to \mathsf{TH}^{\mathscr{M}}_X(Y_0, Y_{-1}, 0) \to 0$$

where $\mathsf{TH}^{\mathscr{M}}_X(Y_0, Y_{-1}, 0)$ is placed in degree 0.

These complexes are functorial. Indeed let $f : Y \to Y'$ be a X-morphism of quasi-projective k-varieties. Let $Y_\bullet$ be a stratification of Y and $Y'_\bullet$ be a stratification of Y' such that $f(Y_i) \subseteq Y'_i$ for every integer $i \in \mathbb{Z}$ (i.e. $Y'_\bullet$ is finer than the image $f_\sharp(Y_\bullet)$ of $Y_\bullet$ by f). Then by Lemma 3.4, the morphisms

$$f^{\mathscr{M}}_\star : \mathsf{TH}^{\mathscr{M}}_X(Y_i, Y_{i-1}, i) \to \mathsf{TH}^{\mathscr{M}}_X(Y'_i, Y'_{i-1}, i)$$

(see §3.3) define a morphism of complexes

$$f^{\mathscr{M}}_\star : \mathsf{TH}^{\mathscr{M}}_X(Y, Y_\bullet) \to \mathsf{TH}^{\mathscr{M}}_X(Y', Y'_\bullet).$$

In particular we have a morphism of complexes

$$f^{\mathscr{M}}_\star : \mathsf{TH}^{\mathscr{M}}_X(Y, Y_\bullet) \to \mathsf{TH}^{\mathscr{M}}_X(Y', f_\sharp(Y_\bullet))$$

and for every morphism $f' : Y' \to Y''$ of quasi-projective k-schemes, the diagram

$$\begin{tikzcd} \mathsf{TH}^{\mathscr{M}}_X(Y, Y_\bullet) \arrow[r, "f^{\mathscr{M}}_\star"'] \arrow[rr, bend left, "(f'f)^{\mathscr{M}}_\star"] & \mathsf{TH}^{\mathscr{M}}_X(Y', f_\sharp(Y_\bullet)) \arrow[r, "(f')^{\mathscr{M}}_\star"'] & \mathsf{TH}^{\mathscr{M}}_X(Y'', (f'f)_\sharp(Y_\bullet)) \end{tikzcd}$$

is commutative.

Definition 4.9. — We define $\mathsf{r}^{\mathscr{M}}_X(Y, Y_\bullet)$ by

$$\mathsf{r}^{\mathscr{M}}_X(Y, Y_\bullet) := \mathsf{TH}^{\mathscr{M}}_X(Y, Y_\bullet)[-2d].$$

The next proposition shows that complexes associated with cellular stratifications do compute the $\mathscr{M}$-homology of quasi-projective k-schemes.

Proposition 4.10. — *Assume that $Y_\bullet$ is a cellular stratification of Y. For every integer $i \in \mathbb{Z}$, there is an isomorphism*

$$\phi(Y, Y_\bullet, i) : \mathsf{H}_i^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet)) \xrightarrow{\sim} \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, i)$$

such that for $Y_\bullet \leqslant Y'_\bullet$ the diagram is commutative:

$$\begin{tikzcd}
\mathsf{H}_i^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet)) \arrow[dd] \arrow[dr, "\phi(Y,Y_\bullet,i)"] & \\
& \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, i) \\
\mathsf{H}_i^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Y, Y'_\bullet)) \arrow[ur, "\phi(Y,Y'_\bullet,i)"'] &
\end{tikzcd}$$

where the vertical morphism is the functoriality morphism.

Proof. — Let n be an integer such that $Y_n = Y$. Let us construct the isomorphisms $\phi(Y, Y_\bullet, i)$ by induction on n. If $n = 0$ then $\mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, i) = 0$ for every integer $i \neq 0$ and the Lemma is obvious. Assume $n = 1$. Using the long exact sequence from Lemma 3.3, Definition 4.1 and Remark 4.3, we obtain the short exact sequence

$$0 \to \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, 1) \to \mathsf{TH}_X^{\mathscr{M}}(Y, Y_0, 1) \to \mathsf{TH}_X^{\mathscr{M}}(Y_0, \emptyset, 0) \to \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, 0) \to 0$$

which proves the Lemma in that case. Assume $n \geqslant 2$. Let $Z = Y_{n-1}$ and

$$Z_\bullet : \emptyset = Z_{-1} \subseteq Z_0 = Y_0 \subseteq Z_1 = Y_1 \subseteq \cdots \subseteq Z_{n-1} = Y_{n-1} = Z$$

be the induced stratification. If $i < 0$ or $i > n$ we set $\phi(Y, Y_\bullet, i) = 0$ which is an isomorphism since $\mathsf{H}_i^{\mathscr{M}}(\mathsf{R}_X^{\mathscr{M}}(Y, Y_\bullet)) = \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, i) = 0$ by Remark 4.3. Let $0 \leqslant i \leqslant n-2$. We have by induction an isomorphism

$$\mathsf{H}_i^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet)) = \mathsf{H}_i^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Z, Z_\bullet)) \xrightarrow{\phi(Z,Z_\bullet,i)} \mathsf{TH}_X^{\mathscr{M}}(Z, \emptyset, i) = \mathsf{TH}_X^{\mathscr{M}}(Y_{n-1}, \emptyset, i)$$

and we let $\phi(Y, Y_\bullet, i)$ be the composition of this isomorphism and the canonical morphism $\mathsf{TH}_X^{\mathscr{M}}(Y_{n-1}, \emptyset, i) \to \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, i)$ which is an isomorphism by Remark 4.2.

Now we have a commutative diagram

$$\begin{tikzcd}
\mathsf{TH}_X^{\mathscr{M}}(Y_n, Y_{n-1}, n) \arrow[r, "\partial_n"] \arrow[dr] \arrow[drr, bend right] & \mathsf{TH}_X^{\mathscr{M}}(Y_{n-1}, Y_{n-2}, n-1) \arrow[r, "\partial_{n-1}"] & \mathsf{TH}_X^{\mathscr{M}}(Y_{n-2}, Y_{n-3}, n-2) \\
& \mathrm{Ker}(\partial_{n-1}) = \mathsf{H}_{n-1}^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Z, Z^\bullet)) \arrow[u] \arrow[r, "\phi(Z,Z_\bullet,n-1)"] & \mathsf{TH}_X^{\mathscr{M}}(Y_{n-1}, \emptyset, n-1)
\end{tikzcd} \tag{12}$$

where the morphism (12) is the morphism in the long exact sequence

$$\begin{tikzcd}
\mathsf{TH}^{\mathscr{M}}_X(Y_{n-1},\emptyset,n) \arrow[d] & & \\
\mathsf{TH}^{\mathscr{M}}_X(Y,\emptyset,n) \arrow[r] & \mathsf{TH}^{\mathscr{M}}_X(Y_n,Y_{n-1},n) \arrow[r, "(12)"] & \mathsf{TH}^{\mathscr{M}}_X(Y_{n-1},\emptyset,n-1) \arrow[d] \\
& & \mathsf{TH}^{\mathscr{M}}_X(Y,\emptyset,n-1) \arrow[d] \\
& & \mathsf{TH}^{\mathscr{M}}_X(Y_n,Y_{n-1},n-1)
\end{tikzcd}$$

obtained by Lemma 3.3. However $\mathsf{TH}^{\mathscr{M}}_X(Y_n,Y_{n-1},n-1)=0$, by Definition 4.1, and $\mathsf{TH}^{\mathscr{M}}_X(Y_{n-1},\emptyset,n)=0$, by Remark 4.3. We obtain therefore an isomorphism

$$\phi(Y,Y_\bullet,n):\mathrm{Ker}(\partial_n)=\mathsf{H}^{\mathscr{M}}_n(\mathsf{TH}^{\mathscr{M}}_X(Y,Y_\bullet))\to\mathsf{TH}^{\mathscr{M}}_X(Y,\emptyset,n)$$

and an isomorphism

$$\phi(Y,Y_\bullet,n-1):\mathsf{H}^{\mathscr{M}}_{n-1}(\mathsf{TH}^{\mathscr{M}}_X(Y,Y_\bullet))\to\mathsf{TH}^{\mathscr{M}}_X(Y,\emptyset,n-1).$$

Hence the statement. □

Remark 4.11. — By Definition 3.1, one may view the isomorphisms constructed in Lemma 4.10, as isomorphisms

$$\mathsf{H}^{2d-i}_{\mathscr{M}}(\mathsf{r}^{\mathscr{M}}_X(Y,Y_\bullet))=\mathsf{H}^{-i}_{\mathscr{M}}(\mathsf{TH}^{\mathscr{M}}_X(Y,Y_\bullet))\xrightarrow{\phi(Y,Y_\bullet,i)}\mathsf{H}^{2d-i}_{\mathscr{M}}(a^{\mathscr{M}}_!a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)))$$

Hence $\psi(Y,Y_\bullet,i):=\phi(Y,Y_\bullet,2d-i)$ are isomorphisms

$$\psi(Y,Y_\bullet,i):\mathsf{H}^i_{\mathscr{M}}(\mathsf{r}^{\mathscr{M}}_X(Y,Y_\bullet))\xrightarrow{\sim}\mathsf{H}^i_{\mathscr{M}}(a^{\mathscr{M}}_!a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X))$$

such that for $Y_\bullet\leqslant Y'_\bullet$ the diagram is commutative:

$$\begin{tikzcd}
\mathsf{H}^i_{\mathscr{M}}(\mathsf{r}^{\mathscr{M}}_X(Y,Y_\bullet)) \arrow[dd] \arrow[dr, "\psi(Y,Y_\bullet,i)"] & \\
& \mathsf{H}^i_{\mathscr{M}}(a^{\mathscr{M}}_!a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)) \\
\mathsf{H}^i_{\mathscr{M}}(\mathsf{r}^{\mathscr{M}}_X(Y,Y'_\bullet)) \arrow[ur, "\psi(Y,Y'_\bullet,i)"'] &
\end{tikzcd}$$

where the vertical morphism is the functoriality morphism.

Corollary 4.12. — *For $Y_\bullet\leqslant Y'_\bullet$ the canonical map*

$$\mathsf{r}^{\mathscr{M}}_X(Y,Y_\bullet)\to\mathsf{r}^{\mathscr{M}}_X(Y,Y'_\bullet)$$

is a quasi-isomorphism in $\mathsf{Ch}^{\mathrm{b}}(\mathscr{M}(X))$.

In the Hodge or perverse case, the result can be strenghtened:

Proposition 4.13. — *Assume $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. Let $a : Y \to X$ be an affine morphism. Assume that Y is smooth of pure dimension n. Then there exists a basic stratification $Y_\bullet$ of Y such that $\mathsf{r}_X^{\mathscr{M}}(Y, Y_\bullet)$ is isomorphic in $\mathrm{D^b}(\mathscr{M}(X))$ to*

$$a_!^{\mathscr{M}} a_{\mathscr{M}}^! \mathbb{Q}_X^{\mathscr{M}}.$$

Proof. — Let r be an integer $0 \leqslant r \leqslant n$. Assume that $Z \subseteq Y$ is a closed subset such that $\dim(Z) \leqslant r$ and

$$\mathsf{H}^i_{\mathscr{M}}(z^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})) = 0$$

for every integer $i \neq 2d - r$. Let $z : Z \hookrightarrow Y$ be the closed immersion. Consider the objet

$$A := \mathsf{H}^{2d-r}_{\mathscr{M}}(z^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})) = z^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})[2d - r]$$

in $\mathscr{M}(Z)$. By [**7**, Lemma 3.3], there exists a dense affine open subscheme U in Z such that the open immersion $u : U \hookrightarrow Z$ satisfies

- the canonical morphism $A \to u_*^{\mathscr{M}} u^*_{\mathscr{M}} A$ is injective;
- for every $i \in \mathbb{Z} \setminus \{0\}$, one has

$$^pH^i a_!^{\mathscr{M}} z_!^{\mathscr{M}} \left(u_*^{\mathscr{M}} u^*_{\mathscr{M}} A\right) = {}^pH^i(a \circ z)_!^{\mathscr{M}} \left(u_*^{\mathscr{M}} u^*_{\mathscr{M}} A\right) = 0.$$

Consider the distinguished triangle in $\mathrm{D^b}(\mathscr{M}(X))$

$$w_!^{\mathscr{M}} w^!_{\mathscr{M}}(A) \to A \to u_*^{\mathscr{M}} u^*_{\mathscr{M}}(A) \xrightarrow{+1}$$

where $w : W \hookrightarrow Z$ is the closed immersion of the complement of U in Z. Note that $\dim(W) \leqslant r - 1$ and

$$w_!^{\mathscr{M}} w^!_{\mathscr{M}}(A) = w_!^{\mathscr{M}} w^!_{\mathscr{M}} z^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})[2d - r].$$

Since $A \to u_*^{\mathscr{M}} u^*_{\mathscr{M}} A$ is an injective morphism of objects in $\mathscr{M}(Z)$, we have

$$w_!^{\mathscr{M}} \mathsf{H}^i_{\mathscr{M}}(w^!_{\mathscr{M}}(A)) = \mathsf{H}^i_{\mathscr{M}}(w_!^{\mathscr{M}} w^!_{\mathscr{M}}(A)) = 0$$

for $i \neq 1$. This implies that $\mathsf{H}^1_{\mathscr{M}}(w^!_{\mathscr{M}}(A)) = w^!_{\mathscr{M}} z^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})[2d + 1 - r]$ belongs to $\mathscr{M}(Z)$ and

$$\operatorname{Coker}\left[A \to u_*^{\mathscr{M}} u^*_{\mathscr{M}} A\right] = \mathsf{H}^1_{\mathscr{M}}(w^!_{\mathscr{M}}(A)) = w^!_{\mathscr{M}} z^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})[2d + 1 - r].$$

Since a is a smooth morphism, one has $a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}) = \mathbb{Q}_Y^{\mathscr{M}}(n - d)[2n - 2d]$. Hence

$$A := a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})[2d - n]$$

belongs to $\mathscr{M}(X)$, since Y is smooth over k of dimension n. Using the above considerations, we construct simultaneously by induction, an acyclic resolution $A^\bullet$ of A for the left exact functor ${}^pH^0 a_!^{\mathscr{M}}$ and a cellular stratification $Y_\bullet$ of Y such that

$$\mathsf{H}^j_{\mathscr{M}}((y_i)^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})) = 0$$

for every integer $j \in \mathbb{Z} \setminus \{2d - i\}$. The resolution is given in terms of the stratification by

$$A^i = (y_{n-i})_!^{\mathscr{M}} (u_{n-i})_*^{\mathscr{M}} (u_{n-i})^*_{\mathscr{M}}((y_{n-i})^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}))[2d + i - n]$$

and

$$\mathrm{Coker}\left[A^i \to A^{i+1}\right] = (y_{n-i-1})_!^{\mathscr{M}}((y_{n-i-1})^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}))[2d+i+1-n]$$

where $u_i : Y_i \backslash Y_{i-1} \hookrightarrow Y_i$ is the open immersion and $y_i : Y_i \hookrightarrow Y$ the closed immersion. Since the resolution is acyclic for the left exact functor ${}^pH^0 a_!^{\mathscr{M}}$ there is an isomorphism in $\mathrm{D^b}(\mathscr{M}(X))$ between $a_!^{\mathscr{M}} A = a_!^{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})[2d-n]$ and the complex

$$\cdots \to 0 \to {}^pH^0 a_!^{\mathscr{M}}(A^0) \to {}^pH^0 a_!^{\mathscr{M}}(A^1) \to \cdots \to {}^pH^0 a_!^{\mathscr{M}}(A^n) \to 0 \to \cdots \qquad (12)$$

where ${}^pH^0 a_!^{\mathscr{M}}(A^0)$ is in degree 0. By construction

$${}^pH^0 a_!^{\mathscr{M}}(A^i) = \mathsf{TH}_X^{\mathscr{M}}(Y_{n-i}, Y_{n-i-1}, n-i)$$

and the complex (12) is nothing but $\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet)[-2d]$. Hence $a_!^{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})$ is isomorphic to $\mathsf{r}_X^{\mathscr{M}}(Y, Y_\bullet)$. This concludes the proof of the Proposition. □

5. Tools from homotopical algebra

Let X be a smooth quasi-projective k-scheme. To construct a realization functor from the category of étale constructible motives, it is handy to have it first defined on the «big category» $\mathbf{DA}^{\text{ét}}(X, \mathbb{Q})$ (it may also be useful in some instances to have such a «big realization»). However, for this, the bounded derived category $\mathrm{D^b}(\mathscr{M}(X))$ of mixed Hodge modules is too small.

In this section, we elaborate on the Gabriel-Quillen embedding theorem (see [**41**, Appendix A] for a very detailed treatment), to explain how one can remedy this problem and embed the bounded derived category into the homotopy category of some stable model category that does the job.

We also prove the results of homotopical algebra needed to construct the realization functors in particular Proposition 5.17 that allows to lift certain functors defined on Sm/X to a Quillen adjunction on the category of presheaves $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$.

In this section, $\mathscr{A}$ is a $\mathbb{Q}$-linear Abelian category. We denote by $0_{\mathscr{A}}$ the zero object in $\mathscr{A}$.

5.1. Let $\mathbf{PSh}(\mathscr{A}, \mathbb{Q})$ be the category of presheaves of $\mathbb{Q}$-vector spaces on $\mathscr{A}$. Since $\mathscr{A}$ is $\mathbb{Q}$-linear, we have the Yoneda functor

$$\begin{array}{rcl} \mathsf{i} : \mathscr{A} & \to & \mathbf{PSh}(\mathscr{A}, \mathbb{Q}) \\ A & \mapsto & \mathrm{Hom}_{\mathscr{A}}(-, A). \end{array}$$

Given an object A in $\mathscr{A}$, we denote by $\mathbb{Q}[A]$ be the free presheaf of $\mathbb{Q}$-vector spaces associated with A: its section on $B \in \mathscr{A}$ are given by the free $\mathbb{Q}$-vector space $\mathbb{Q}[\mathrm{Hom}_{\mathscr{A}}(B, A)]$ on the set $\mathrm{Hom}_{\mathscr{A}}(B, A)$.

Denote by $\mathbf{PSha}(\mathscr{A}, \mathbb{Q})$ the full subcategory of $\mathbf{PSh}(\mathscr{A}, \mathbb{Q})$ with objects the additive presheaves of $\mathbb{Q}$-vector spaces (or equivalently the $\mathbb{Q}$-linear presheaves). The forgetful functor admits a left adjoint

$$a_{\mathrm{ad}} : \mathbf{PSh}(\mathscr{A}, \mathbb{Q}) \to \mathbf{PSha}(\mathscr{A}, \mathbb{Q})$$

given, for a $\mathscr{F} \in \mathbf{PSh}(\mathscr{A}, \mathbb{Q})$, by the colimit $a_{\mathrm{ad}}(\mathscr{F}) := \mathrm{colim}_{(\mathsf{i}(A)\to\mathscr{F})\mathsf{i}\downarrow\mathscr{F}}\, \mathsf{i}(A)$ where $\mathsf{i} \downarrow \mathscr{F}$ is the over category in the category of presheaves of sets (in other words $\mathscr{F}$ is viewed as a presheaf of sets and i as functor from $\mathscr{A}$ to presheaves of sets on $\mathscr{A}$).

Remark 5.1. — Note that if $A \in \mathscr{A}$ and $\mathscr{F} \in \mathbf{PSh}(\mathscr{A}, \mathbb{Q})$, then is it not true in general that the canonical injection

$$\mathrm{Hom}_{\mathbf{PSh}(\mathscr{A},\mathbb{Q})}(\mathsf{i}(A), \mathscr{F}) \hookrightarrow \mathscr{F}(A)$$

is an isomorphism. This is however true as soon as $\mathscr{F}$ is additive.

Remark 5.2. — Note that the presheaves of the form $\mathbb{Q}[A]$ are never additive, indeed $\mathbb{Q}[A](0_{\mathscr{A}})$ is equal to $\mathbb{Q}$ and not to zero as it should be for an additive presheaf. In fact, we have a canonical isomorphism

$$a_{\mathrm{ad}}(\mathbb{Q}[A]) = \mathrm{Hom}_{\mathscr{A}}(-, A) = \mathsf{i}(A).$$

Indeed, for every $\mathscr{F} \in \mathbf{PSha}(\mathscr{A}, \mathbb{Q})$, we have by Remark 5.1, isomorphisms functorial in $\mathscr{F}$

$$\begin{aligned}\mathrm{Hom}_{\mathbf{PSha}(\mathscr{A},\mathbb{Q})}(\mathsf{i}(A), \mathscr{F}) &= \mathscr{F}(A) = \mathrm{Hom}_{\mathbf{PSh}(\mathscr{A},\mathbb{Q})}(\mathbb{Q}[A], \mathscr{F})\\ &= \mathrm{Hom}_{\mathbf{PSha}(\mathscr{A},\mathbb{Q})}(a_{\mathrm{ad}}(\mathbb{Q}[A]), \mathscr{F}).\end{aligned}$$

5.2. Consider the Grothendieck pretopology on $\mathscr{A}$ (see [**1**, Exposé II, Définition 1.3]) such that covering families of an object $A \in \mathscr{A}$ are families with one element $\{a : B \twoheadrightarrow A\}$ where a is an epimorphism and let $\mathbf{Sh}(\mathscr{A}, \mathbb{Q})$ be the category of sheaves of $\mathbb{Q}$-vector spaces for this topology. A presheaf $\mathscr{F} \in \mathbf{PSh}(\mathscr{A}, \mathbb{Q})$ is a sheaf if and only if for every epimorphism $a : B \twoheadrightarrow A$ the sequence

$$0 \to \mathscr{F}(A) \to \mathscr{F}(B) \to \mathscr{F}(B \times_A B)$$

is exact and the objects in

$$\mathbf{Sha}(\mathscr{A}, \mathbb{Q}) := \mathbf{PSha}(\mathscr{A}, \mathbb{Q}) \cap \mathbf{Sh}(\mathscr{A}, \mathbb{Q})$$

are precisely the left exact $\mathbb{Q}$-linear contravariant functors from $\mathscr{A}$ to the category of $\mathbb{Q}$-vector spaces.

We have the sheafification functor

$$a_{\mathrm{epi}} : \mathbf{PSh}(\mathscr{A}, \mathbb{Q}) \to \mathbf{Sh}(\mathscr{A}, \mathbb{Q}).$$

Let us recall briefly its construction (see e.g. [**41**, §A.7.8]). For $A \in \mathscr{A}$, let $\mathcal{C}_A$ be the following filtered category. The objects in $\mathcal{C}_A$ are epimorphisms $B \twoheadrightarrow A$. Between two objects there is at most one map. There exists a map $(b : B \twoheadrightarrow A) \to (b' : B' \twoheadrightarrow A)$ if and only if there is a map $b'' : B' \to B$ such that $b \circ b'' = b'$. Given a presheaf $\mathscr{F} \in \mathbf{PSh}(\mathscr{A}, \mathbb{Q})$, sending an object $B \twoheadrightarrow A$ to $\mathrm{Ker}(\mathscr{F}(B) \to \mathscr{F}(B \times_A B))$ is a functor from the filtered category $\mathcal{C}_A$ to the category of $\mathbb{Q}$-vector spaces. One defines then

$$\mathrm{L}\mathscr{F}(A) := \operatorname*{colim}_{(B\twoheadrightarrow A)\in\mathcal{C}_A} \mathrm{Ker}(\mathscr{F}(B) \to \mathscr{F}(B \times_A B))$$

and $a_{\mathrm{epi}}\mathscr{F} = \mathrm{LL}\mathscr{F}$.

Remark 5.3. — Given $\mathscr{F} \in \mathbf{PSh}(\mathscr{A}, \mathbb{Q})$, recall that $\mathrm{L}\mathscr{F} = 0$ if and only for every for every $A \in \mathscr{A}$ and every $\alpha \in \mathscr{F}(A)$, there exists an epimorphism $b : B \twoheadrightarrow A$ in $\mathscr{A}$ such that $b^*\alpha = 0$ in $\mathscr{F}(B)$ (see [**41**, A.7.11. Lemma]).

In particular, given a sequence $\mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$ in $\mathbf{PSh}(\mathscr{A}, \mathbb{Q})$, the sequence

$$a_{\mathrm{epi}}\mathscr{F}' \xrightarrow{a_{\mathrm{epi}}\phi} a_{\mathrm{epi}}\mathscr{F} \xrightarrow{a_{\mathrm{epi}}\psi} a_{\mathrm{epi}}\mathscr{F}''$$

is exact in $\mathbf{Sh}(\mathscr{A}, \mathbb{Q})$ if and only if for every $A \in \mathscr{A}$ and every $\alpha \in \mathscr{F}(A)$ such that $\psi(\alpha) = 0$ in $\mathscr{F}''(A)$ there exists an epimorphism $b : B \to A$ in $\mathscr{A}$ and an element $\beta \in \mathscr{F}'(B)$ such that $\phi(\beta) = b^*\alpha$.

If $\mathscr{F}$ is additive then $a_{\mathrm{epi}}\mathscr{F}$ is also additive (see [**41**, §A.7.8]), hence the functor a_{epi} induce a functor

$$a_{\mathrm{epi}} : \mathbf{PSha}(\mathscr{A}, \mathbb{Q}) \to \mathbf{Sha}(\mathscr{A}, \mathbb{Q}) \tag{13}$$

which is left adjoint to the forgetful functor.

Lemma 5.4. — *The category $\mathbf{Sha}(\mathscr{A}, \mathbb{Q})$ is a $\mathbb{Q}$-linear Abelian category. The Yoneda functor*

$$\begin{array}{rcl} \mathsf{i} : \mathscr{A} & \to & \mathbf{Sha}(\mathscr{A}, \mathbb{Q}) \\ A & \mapsto & \mathrm{Hom}_{\mathscr{A}}(-, A) \end{array}$$

is a fully faithful exact functor and $\mathscr{A}$ is stable by extension in $\mathbf{Sha}(\mathscr{A}, \mathbb{Q})$. Moreover the induced functor

$$\mathrm{D}^\star(\mathscr{A}) \to \mathrm{D}^\star_{\mathscr{A}}(\mathbf{Sha}(\mathscr{A}, \mathbb{Q})),$$

where $\star \in \{-, \mathrm{b}\}$, is an equivalence of categories.

Remark 5.5. — Let $\mathbf{Sha}(\mathscr{A}, \mathbb{Z})$ be the category of additive sheaves of Abelian groups on $\mathscr{A}$ for the topology of epimorphisms (i.e. the category of additive left exact functors from $\mathscr{A}$ to the category of Abelian groups as considered in [**12**, II §2] and [**35**]). Since $\mathscr{A}$ is $\mathbb{Q}$-linear, the canonical functor $\mathbf{Sha}(\mathscr{A}, \mathbb{Q}) \to \mathbf{Sha}(\mathscr{A}, \mathbb{Z})$ is an exact equivalence of categories. In particular the statement of Lemma 5.4 is simply the embedding theorem proved by P. Gabriel in [**12**] and generalized to exact categories by D. Quillen in [**35**].

5.3. Recall that the category $\mathsf{Ch}(\mathbb{Q})$ of cochain complexes of $\mathbb{Q}$-vector spaces has a model structure (called the projective model structure) such that the weak equivalences are the quasi-isomorphisms and the fibrations are the epimorphisms (see [**17**, Theorem 2.3.11]).

Notation 5.6. — Let $\mathscr{B}$ be an Abelian category. Given $B \in \mathscr{B}$ and an integer $n \in \mathbb{Z}$, we denote by $S^n(B)$ the complex concentrated in degree n with $S^n(B)^n = B$ and $D^n(B)$ be the complex concentrated in degree n, $n+1$ with $D^n(B)^n = D^n(B)^{n+1} = B$ and the identity as only non zero differential. Note that the identity induces a map $S^{n+1}(B) \to D^n(B)$.

Let I be the set of maps $S^{n+1}(\mathbb{Q}) \to D^n(\mathbb{Q})$ and J be the set of maps $0 \to D^n(\mathbb{Q})$. The projective model structure on $\mathsf{Ch}(\mathbb{Q})$ is cofibrantly generated. The set I (resp. J) is a set of generating cofibrations (resp. trivial cofibrations). In other words $\mathsf{Fib} = \mathrm{RLP}(J)$ and $\mathsf{Fib} \cap \mathsf{W} = \mathrm{RLP}(I)$

5.4. We consider the projective model structure on the category $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ of presheaves of $\mathbb{Q}$-vector spaces on $\mathscr{A}$: the fibrations (resp. equivalences) are the maps of presheaves of complexes $\mathscr{F} \to \mathscr{G}$ such that $\mathscr{F}(A) \to \mathscr{G}(A)$ is a fibration (resp. an equivalence) in $\mathsf{Ch}(\mathbb{Q})$ for every $A \in \mathscr{A}$. This model structure is cofibrantly generated: the maps $S^{n+1}(\mathbb{Q}[A]) \to D^n(\mathbb{Q}[A])$, with $A \in \mathscr{A}$, form a class of generating cofibrations $I_{\mathscr{A}}$ and and the maps $0 \to D^n(\mathbb{Q}[A])$, with $A \in \mathscr{A}$, form a class of generating trivial cofibrations $J_{\mathscr{A}}$.

On the category $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$, we consider the model structure provided by the following Lemma (and called in the sequel projective model structure):

Lemma 5.7. — *Let* W, Fib *be the class of maps in* $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ *defined as follows: a map belongs to* W *(resp.* Fib*) if and only if it is an equivalence (resp. a projective fibration) in* $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$*. Let* Cof *be the class of maps in* $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ *that have the left lifting property with respect to maps in* $\mathsf{W} \cap \mathsf{Fib}$*. Then the triplet* $(\mathsf{W}, \mathsf{Fib}, \mathsf{Cof})$ *defines a model structure on* $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$*.*

Proof. — Note that the class of maps in $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ which are monomorphisms and quasi-isomorphisms is stable by pushouts, transfinite compositions and retracts. By Remark 5.2, the class $a_{\mathrm{ad}}(J_{\mathscr{A}})$ consists of the morphisms $0 \to D^n(\mathsf{i}(A))$ with $A \in \mathscr{A}$ and $n \in \mathbb{Z}$ which are all monomorphisms and quasi-isomorphisms. Hence every relative $a_{\mathrm{ad}}(J_{\mathscr{A}})$-cell complex is a quasi-isomorphism in $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. The Lemma follows then from [**16**, Theorem 11.3.2] (see also [**10**, Theorem 3.3]). □

Remark 5.8. — The class $a_{\mathrm{ad}}(I_{\mathscr{A}})$ (resp. $a_{\mathrm{ad}}(J_{\mathscr{A}})$) are generating cofibrations (resp. trivial cofibrations) for the projective model structure of Lemma 5.7. In particular since all the maps in $a_{\mathrm{ad}}(I_{\mathscr{A}})$ are monomorphisms and monomorphisms are stable by pushouts, retracts and transfinite compositions, it follows that all cofibrations are monomorphisms.

Remark 5.9. — The image by i of bounded complexes of objects in $\mathscr{A}$ are cofibrant for the projective model structure on $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$.

5.5. Now let us endow the category $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ with its τ-local projective model structure where τ is the topology of epimorphisms (i.e. we consider the left Bousfield localization of the projective model structure of Lemma 5.7 with respect to the maps that induce quasi-isomorphims on the associated complexes of sheaves). Let us consider the full subcategory $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ formed by the additive presheaves of complexes of $\mathbb{Q}$-vector spaces. The functor (13) induce a functor

$$a_{\mathrm{epi}} : \mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \to \mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$$

left afjoint to the forgetful functor.

Consider the class W, Fib of maps in $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ defined as follows. A map $\mathscr{F} \to \mathscr{G}$ belongs to W (resp. Fib) if and only if it is a τ-local weak equivalence (resp. a τ-local fibration) in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. Let Cof be the class of maps in $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ that have the left lifting property with respect to maps in $\mathsf{W} \cap \mathsf{Fib}$.

By [**3**, Lemme 4.4.41], the triplet $(\mathsf{W}, \mathsf{Fib}, \mathsf{Cof})$ is a model structure (called the projective model structure) on the category $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) = \mathsf{Ch}(\mathbf{Sha}(\mathscr{A}, \mathbb{Q}))$ and we have a Quillen adjunction

$$a_{\mathrm{epi}} : \mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \leftrightarrows \mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$$

for the projective model structures. Note that since (13) is an exact functor, the left adjoint preserves equivalences (i.e. quasi-isomorphisms).

Remark 5.10. — Note that $\mathbf{Sha}(\mathscr{A}, \mathbb{Q})$ is an Abelian category and the weak equivalence for the above model structure are the quasi-isomorphisms. In particular

$$\mathsf{Ho}(\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))) = \mathrm{D}(\mathbf{Sha}(\mathscr{A}, \mathbb{Q})).$$

5.6. The category $\mathsf{Ch}(\mathbb{Q})$ with its projective model structure is symmetric monoidal model category for the usual tensor product, denoted by $\otimes_{\mathsf{Ch}}$, of complexes of $\mathbb{Q}$-vector spaces (see e.g. [**17**, Proposition 4.2.13]). The category of presheaves $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ with its projective model structure is symmetric monoidal model category (see e.g. [**3**, Proposition 4.4.63]), the tensor product $\mathscr{F} \otimes \mathscr{G}$ of two objects $\mathscr{F}, \mathscr{G} \in \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ being the presheaf on $\mathscr{A}$

$$A \mapsto \mathscr{F}(A) \otimes_{\mathsf{Ch}} \mathscr{G}(A).$$

Given a complex of $\mathbb{Q}$-vector spaces K let K_{cst} be the constant presheaf of $\mathbb{Q}$-vector spaces on $\mathscr{A}$. Given an object $\mathscr{F} \in \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$, the functor

$$\begin{aligned} \mathscr{F} \otimes (-)_{cst} : \mathsf{Ch}(\mathbb{Q}) &\to \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \\ K &\mapsto \mathscr{F} \otimes K_{cst} \end{aligned} \tag{14}$$

has a right adjoint

$$\begin{aligned} \underline{\mathrm{Hom}}(\mathscr{F}, -) : \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) &\to \mathsf{Ch}((\mathbb{Q}) \\ \mathscr{G} &\mapsto \underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G}) \end{aligned} \tag{15}$$

For every integer n, the elements in $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})^n$ are the graded morphisms of complexes $\mathscr{F} \to \mathscr{G}$ of degree n.

Remark 5.11. — If $\mathscr{G} \in \mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$, then $\underline{\mathrm{Hom}}(a_{\mathrm{ad}}(\mathscr{F}), \mathscr{G}) = \underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})$.

Note that we have the Quillen adjunction for the projective model structures

$$(-)_{cst} : \mathsf{Ch}(\mathbb{Q}) \rightleftarrows \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) : \Gamma(0_{\mathscr{A}}, -)$$

this implies that the bifunctor

$$- \otimes (-)_{cst} : \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \times \mathsf{Ch}(\mathbb{Q}) \to \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \tag{16}$$

is a Quillen bifunctor for the projective model structures. If $\mathscr{F} \in \mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$, then $\mathscr{F} \otimes K_{cst}$ belongs also to $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. In particular if $\mathscr{F} \in \mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ then (14) and (15) induce adjoint functors

$$\mathscr{F} \otimes (-)_{cst} : \mathsf{Ch}(\mathbb{Q}) \rightleftarrows \mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) : \underline{\mathrm{Hom}}(\mathscr{F}, -). \tag{17}$$

Remark 5.12. — For every $\mathscr{F} \in \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ we have

$$a_{\mathrm{ad}}(\mathscr{F} \otimes K_{cst}) = a_{\mathrm{ad}}(\mathscr{F} \otimes K_{cst}.$$

Indeed for every $\mathscr{G} \in \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$, using Remark 5.11

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))}(a_{\mathrm{ad}}(\mathscr{F} \otimes K_{cst}), \mathscr{G}) &= \mathrm{Hom}_{\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))}(\mathscr{F} \otimes K_{cst}, \mathscr{G}) \\
&= \mathrm{Hom}_{\mathsf{Ch}(\mathbb{Q})}(K, \underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})) \\
&= \mathrm{Hom}_{\mathsf{Ch}(\mathbb{Q})}(K, \underline{\mathrm{Hom}}(a_{\mathrm{ad}}(\mathscr{F}), \mathscr{G})) \\
&= \mathrm{Hom}_{\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))}(a_{\mathrm{ad}}(\mathscr{F}) \otimes K_{cst}, \mathscr{G})
\end{aligned}$$

Lemma 5.13. — *The bifunctor*

$$- \otimes (-)_{cst} : \mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \times \mathsf{Ch}(\mathbb{Q}) \to \mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$$

is a Quillen bifunctor for the projective model structures.

Proof. — Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ and $u : K \to L$ be a morphism in $\mathsf{Ch}(\mathbb{Q})$. Let $\mathscr{H}$ be the pushout in the category $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ of the diagram

$$\begin{tikzcd}
\mathscr{F} \otimes K_{cst} \arrow[r, "\mathscr{F} \otimes u_{cst}"] \arrow[d, "f \otimes K_{cst}"'] & \mathscr{F} \otimes L_{cst} \\
\mathscr{G} \otimes K_{cst}. &
\end{tikzcd} \tag{18}$$

We have to prove that if f and a are cofibrations in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$, then the map

$$\mathscr{H} \to \mathscr{G} \otimes L_{cst} \tag{19}$$

is a cofibration in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ which is trivial is either f or u are trivial cofibrations. Assume that f is the image under a_{ad} of a cofibration $f' : \mathscr{F}' \to \mathscr{G}'$ in the category $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. Let $\mathscr{H}'$ be the pushout in $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ of the diagram similar to (18) obtained by replacing f by f'. Since (16) is a Quillen bifunctor for the projective model structures, the map $\mathscr{H}' \to \mathscr{G}' \otimes L_{cst}$ is a cofibration in $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})$ which is trivial if a is a trivial cofibration in $\mathsf{Ch}(\mathbb{Q})$ or f' is a trivial cofibration in $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. This implies that its image (19) under a_{ad} is a cofibration in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})$ which is trivial if a is a trivial cofibration in $\mathsf{Ch}(\mathbb{Q})$ or f' is a trivial cofibration in $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$.

Since $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ is cofibrantly generated, with $a_{\mathrm{ad}}(I_{\mathscr{A}})$ and $a_{\mathrm{ad}}(J_{\mathscr{A}})$ as generating cofibrations and trivial cofibrations, the lemma follows from the above case and [**17**, Corollary 4.2.5]. □

Lemma 5.14. — *The bifunctor*

$$-\otimes(-)_{cst} : \mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \times \mathsf{Ch}(\mathbb{Q}) \to \mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$$

is a Quillen bifunctor for the projective model structures.

Proof. — Since every cofibration in $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ is the image under a_{epi} of a τ-local projective cofibration in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ by [**3**, Lemme 4.4.41], it is enough to prove that $-\otimes(-)_{cst}$ is a Quillen bifunctor for the τ-local projective model structure on $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ and the projective model structure on $\mathsf{Ch}(\mathbb{Q})$. Let $f : \mathscr{F} \to \mathscr{G}$ be a τ-local cofibration in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ and $u : K \to L$ be a cofibration in $\mathsf{Ch}(\mathbb{Q})$. Let $\mathscr{H}$ be the pushout in the category $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ of the diagram

$$\begin{tikzcd} \mathscr{F}\otimes K_{cst} \arrow[r, "\mathscr{F}\otimes u_{cst}"] \arrow[d, "f\otimes K_{cst}"'] & \mathscr{F}\otimes L_{cst} \\ \mathscr{G}\otimes K_{cst}. & \end{tikzcd} \tag{20}$$

Since the τ-local model structure is obtained by a Bousfield localization, the τ-local cofibration are the projective cofibrations and it follows from Lemma 5.13 that

$$\mathscr{H} \to \mathscr{G}\otimes L_{cst} \tag{21}$$

is a τ-local cofibration in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ which is trivial if u is a trivial cofibration. Assume that f is also a τ-local weak equivalence. Since f is a cofibration it is also a monomorphism (see Remark 5.8), and therefore $a_{\mathrm{epi}}(f)$ is a monomorphism and a quasi-isomorphism. The square

$$\begin{tikzcd} a_{\mathrm{epi}}(\mathscr{F})\otimes K_{cst} \arrow[r, "a_{\mathrm{epi}}(\mathscr{F})\otimes u_{cst}"] \arrow[d, "a_{\mathrm{epi}}(f)\otimes K_{cst}"'] & a_{\mathrm{epi}}(\mathscr{F})\otimes L_{cst} \arrow[d] \\ a_{\mathrm{epi}}(\mathscr{G})\otimes K_{cst} \arrow[r] & a_{\mathrm{epi}}(\mathscr{H}) \end{tikzcd}$$

being a pushout square, it follows that the map $a_{\mathrm{epi}}(\mathscr{F})\otimes L_{cst} \to a_{\mathrm{epi}}(\mathscr{H})$ is a quasi-isomorphism. The composition

$$a_{\mathrm{epi}}(\mathscr{F})\otimes L_{cst} \to a_{\mathrm{epi}}(\mathscr{H}) \xrightarrow{a_{\mathrm{epi}}((21))} a_{\mathrm{epi}}(\mathscr{G})\otimes L_{cst}$$

being equal to $a_{\mathrm{epi}}(f)\otimes L_{cst}$ which is a quasi-isomorphism, it follows that $a_{\mathrm{epi}}((21))$ is a quasi-isomorphism and therefore (21) is a trivial τ-local cofibration.

□

5.7. Let $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ be the category of simplicial objects in $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. It is tensored and cotensored over the category of simplicial sets $\Delta^{\mathrm{op}}\mathsf{Sets}$. For $\mathscr{F} \in \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ and $S \in \Delta^{\mathrm{op}}\mathsf{Sets}$, the tensor product $\mathscr{F}\odot S$ is defined as the simplicial objects

$$n \mapsto \mathscr{F}_n \odot S_n := \coprod_{s\in\ S_n} \mathscr{F}_n.$$

By [**37**, Proposition 4.5] the category $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ has a simplicial model structure i.e. such that

$$- \odot - : \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \times \Delta^{\mathrm{op}}\mathsf{Sets} \to \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$$

is a Quillen bifunctor. This model structure is called the canonical model structure and obtained from the Reedy model structure. Recall that the Reedy weak equivalences are the level weak equivalences and that a map $\mathscr{F} \to \mathscr{G}$ is called a Reedy cofibration if for every integer n the map

$$\mathscr{F}_n \coprod_{\mathrm{L}_n(\mathscr{F})} \mathrm{L}_n(\mathscr{G}_n) \to \mathscr{G}_n$$

is a cofibration in $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ where $\mathrm{L}_n(-)$ is the n-th latching space functor. The left derived functor of the colimit functor provides a functor

$$\mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}} : \mathsf{Ho}_{\mathrm{Reedy}}(\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))) \to \mathsf{Ho}(\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})))$$

and a map $\mathscr{F} \to \mathscr{G}$ is a canonical equivalence if its image under this functor is an isomorphism. The canonical cofibrations are the Reedy cofibrations and fibrations are defined as maps having the right lifting properties with respect trivial cofibrations

Let $\mathsf{cc}(\mathscr{F})$ be the constant simplicial object and $\mathsf{Ev}(\mathscr{G}) = \mathscr{G}_0$, then the adjoint functors c and Ev provides a Quillen equivalence

$$\mathsf{cc} : \mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \rightleftarrows \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) : \mathsf{Ev} \tag{22}$$

(see [**37**, Theorem 3.6])

Lemma 5.15. — *The bifunctor*

$$- \otimes (-)_{cst} : \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \times \mathsf{Ch}(\mathbb{Q}) \to \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$$

is a Quillen bifunctor where $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ is endowed with the canonical model structure and $\mathsf{Ch}(\mathbb{Q})$ with the projective model structure.

Proof. — Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ and $u : K \to L$ be a morphism in $\mathsf{Ch}(\mathbb{Q})$. Let $\mathscr{H}$ be the pushout in the category $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ of the diagram

$$\begin{tikzcd} \mathscr{F} \otimes K_{cst} \arrow[r, "\mathscr{F}\otimes u_{cst}"] \arrow[d, "f\otimes K_{cst}"'] & \mathscr{F} \otimes L_{cst} \\ \mathscr{G} \otimes K_{cst}. & \end{tikzcd} \tag{23}$$

We have to prove that if u is a projective cofibration in $\mathsf{Ch}(\mathbb{Q})$ and f is a cofibration in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ for the canonical model structure, then the map

$$\mathscr{H} \to \mathscr{G} \otimes L_{cst} \tag{24}$$

is a a canonical cofibration in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ which is trivial is either f or u are trivial cofibrations. Since the latching space functor is a left adjoint, the square

$$\begin{tikzcd} \mathrm{L}_n(\mathscr{F}) \otimes K_{cst} \arrow[r, "\mathrm{L}_n(\mathscr{F})\otimes u_{cst}"] \arrow[d, "f\otimes K_{cst}"'] & \mathrm{L}_n(\mathscr{F}) \otimes L_{cst} \arrow[d] \\ \mathrm{L}_n(\mathscr{G}) \otimes K_{cst} \arrow[r] & \mathrm{L}_n(\mathscr{H}) \end{tikzcd}$$

is a pushout square.

We have to check that

$$\mathscr{H}_n \sqcup_{\mathrm{L}_n(\mathscr{H})} (\mathrm{L}_n(\mathscr{G}) \otimes L_{cst}) \to \mathscr{G}_n \otimes L_{cst} \tag{25}$$

is a cofibration. For this remark that $\mathscr{H}_n \sqcup_{\mathrm{L}_n(\mathscr{H})} (\mathrm{L}_n(\mathscr{G}) \otimes L_{cst})$ is the pushout of the diagram

$$\begin{tikzcd} (\mathscr{F}_n \sqcup_{\mathrm{L}_n(\mathscr{F})} \mathrm{L}_n(\mathscr{G})) \otimes K_{cst} \arrow[r, "\mathrm{Id}\otimes u_{cst}"] \arrow[d, "f\otimes K_{cst}"'] & (\mathscr{F}_n \sqcup_{\mathrm{L}_n(\mathscr{F})} \mathrm{L}_n(\mathscr{G})) \otimes L_{cst} \\ \mathscr{G}_n \otimes K_{cst}. & \end{tikzcd}$$

Since f is a Reedy cofibration, the map $\mathscr{F}_n \sqcup_{\mathrm{L}_n(\mathscr{F})} \mathrm{L}_n(\mathscr{G}) \to \mathscr{G}_n$ is a cofibration and therefore by Lemma 5.14, the map (25) is a cofibration.

Note that since f is a Reedy cofibration, for every $n \in \mathbb{N}$, the induced map $\mathscr{F}_n \to \mathscr{G}_n$ is a cofibration (i.e. Reedy cofibration are also levelwise cofibration [**16**, Proposition 16.3.11]). Hence for every $n \in \mathbb{N}$, $\mathscr{F}_n \to \mathscr{G}_n$ is a monomorphism and therefore $\mathscr{F} \to \mathscr{G}$ is a monomorphism. This implies that we have a short exact sequence

$$0 \to \mathscr{F} \otimes K_{cst} \to (\mathscr{F} \otimes L_{cst}) \oplus (\mathscr{G} \otimes K_{cst}) \to \mathscr{H} \to 0$$

and thus a distinguished triangle in $\mathsf{Ho}_{\mathrm{Reedy}}(\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})))$

$$\mathscr{F} \otimes K_{cst} \to (\mathscr{F} \otimes L_{cst}) \oplus (\mathscr{G} \otimes K_{cst}) \to \mathscr{H} \xrightarrow{+1} .$$

Now since the left derived functor of the colimit functor (see e.g. [**3**, Lemme 4.1.51]) is triangulated, it yields a distinguished triangle in $\mathsf{Ho}(\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})))$

$$\mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathscr{F} \otimes K_{cst} \to \mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathscr{F} \otimes L_{cst}) \oplus \mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G} \otimes K_{cst}) \to \mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathscr{H} \xrightarrow{+1} .$$

Assume that f is a canonical weak equivalence. The map $\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}} \mathscr{F} \otimes K_{cst} \to \mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G} \otimes K_{cst})$ is then an isomorphism. This implies that the map $\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathscr{F} \otimes L_{cst}) \to \mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}} \mathscr{H}$ is also an isomorphism. Since f is a canonical weak equivalence, the composition

$$\mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathscr{F} \otimes L_{cst}) \to \mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathscr{H} \xrightarrow{\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}((24))} \mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G} \otimes L_{cst})$$

is an isomorphism and therefore so is the second map. This shows that (24) is a canonical weak equivalence.

Assume that u is a trivial cofibration, then the map $\mathscr{F} \otimes K_{cst} \to \mathscr{G} \otimes B_{cst}$ is a level weak equivalence and therefore a realization weak equivalence. The map

$\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}\mathscr{F}\otimes K_{cst}\to\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G}\otimes L_{cst})$ is thus an isomorphism. This implies that the map $\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G}\otimes K_{cst})\to\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}\mathscr{H}$. Since $\mathscr{G}\otimes a$ is a canonical weak equivalence, the composition

$$\mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G}\otimes K_{cst})\to\mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}\mathscr{H}\xrightarrow{\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}((24))}\mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G}\otimes L_{cst})$$

is an isomorphism and therefore so is the second map. This shows that (24) is a canonical weak equivalence. □

Let $\mathscr{F}\in\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q}))$. One has then an adjunction

$$\mathscr{F}\otimes(-)_{cst}:\mathsf{Ch}(\mathbb{Q})\rightleftarrows\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q})):\underline{\mathrm{Hom}}(\mathscr{F},-)\tag{26}$$

where, for every $\mathscr{G}$ in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q}))$, the complex $\underline{\mathrm{Hom}}(\mathscr{F},\mathscr{G})$ is the equalizer of the maps

$$\prod_{n\in\Delta^{\mathrm{op}}}\underline{\mathrm{Hom}}(\mathscr{F}_n,\mathscr{G}_n)\rightrightarrows\prod_{n\to m\in\Delta^{\mathrm{op}}}\underline{\mathrm{Hom}}(\mathscr{F}_n,\mathscr{G}_m).$$

As a consequence one gets immediately the following corollary:

Corollary 5.16. — *Let $\mathscr{F}$ be a cofibrant object in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q}))$, then* (26) *is a Quillen adjunction.*

5.8. Let $\mathcal{S}$ be a category and

$$r:\mathcal{S}\to\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q}))$$

be a functor. With this functor are associated two functors

$$r^*:\mathbf{PSh}(\mathcal{S},\mathsf{Ch}(\mathbb{Q}))\leftrightarrows\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q})):r_*$$

defined as follows. Given an object $\mathscr{F}\in\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q}))$, $r_*(\mathscr{F})$ is the presheaf on $\mathcal{S}$ with values in $\mathsf{Ch}(\mathbb{Q})$ defined by

$$r_*(\mathscr{F})(X):=\underline{\mathrm{Hom}}(r(X),\mathscr{F})$$

for $X\in\mathcal{S}$. Given a presheaf $\mathscr{X}\in\mathbf{PSh}(\mathcal{S},\mathsf{Ch}(\mathbb{Q}))$, the object $r^*(\mathscr{X})$ is defined as the coequalizer in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q}))$

$$r^*(\mathscr{X})=\operatorname{Coeq}\left[\bigoplus_{X\to Y\in\mathsf{Fl}(\mathcal{S})}r(X)\otimes\mathscr{X}(Y)_{cst}\rightrightarrows\bigoplus_{X\in\mathcal{S}}r(X)\otimes\mathscr{X}(X)_{cst}\right]\tag{27}$$

Recall that with an object $X\in\mathcal{S}$ and a presheaf $\mathscr{X}\in\mathbf{PSh}(\mathcal{S},\mathsf{Ch}(\mathbb{Q}))$ is associated an object $X\otimes\mathscr{X}\in\mathbf{PSh}(\mathcal{S},\mathsf{Ch}(\mathbb{Q}))$ (see e.g. [**3**, §4.4]). Given an object $K\in\mathsf{Ch}(\mathbb{Q})$, we denote by K_{cst} the constant presheaf on $\mathcal{S}$ with value K.

Proposition 5.17. — *The functors*

$$r^*:\mathbf{PSh}(\mathcal{S},\mathsf{Ch}(\mathbb{Q}))\leftrightarrows\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q})):r_*$$

are adjoint and the functors r and $r^*(-\otimes\mathbb{Q})$, are canonically isomorphic. Moreover if $r(X)$ is cofibrant in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q}))$ for every $X\in\mathcal{S}$, then they form a Quillen adjunction for the projective model structure on $\mathbf{PSh}(\mathcal{S},\mathsf{Ch}(\mathbb{Q}))$.

Proof. — We simply denote by Hom the set of morphisms in the category $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. Then $\mathrm{Hom}_{\mathbf{PSha}}(r^*(\mathscr{X}), \mathscr{F})$ is by definition the equalizer of

$$\prod_{X\in\mathcal{S}} \mathrm{Hom}_{\mathbf{PSha}}(r(X)\otimes\mathscr{X}(X)_{cst}, \mathscr{F}) \rightrightarrows \prod_{X\to Y\in\mathsf{Fl}(\mathcal{S})} \mathrm{Hom}_{\mathbf{PSha}}(r(X)\otimes\mathscr{X}(Y)_{cst}, \mathscr{F})$$

But for objects $U, V \in \mathcal{S}$

$$\begin{aligned}\mathrm{Hom}(r(U)\otimes\mathscr{X}(V)_{cst}, A) &= \mathrm{Hom}_{\mathsf{Ch}(\mathbb{Q})}(\mathscr{X}(V), \underline{\mathrm{Hom}}(r(U), \mathscr{F}))\\ &= \mathrm{Hom}_{\mathsf{Ch}(\mathbb{Q})}(\mathscr{X}(V), r_*(\mathscr{F})(U)).\end{aligned}$$

This means that $\mathrm{Hom}(r^*(\mathscr{X}), \mathscr{F})$ is equal to the set $\mathrm{Hom}_{\mathbf{PSh}(\mathcal{S},\mathsf{Ch}(\mathbb{Q}))}(\mathscr{X}, r_*(\mathscr{F}))$ of morphisms in $\mathbf{PSh}(\mathcal{S}, \mathsf{Ch}(\mathbb{Q}))$.

Assume that $r(X)$ is cofibrant for every $X \in \mathcal{S}$. If $a : \mathscr{F} \to \mathscr{G}$ is fibration (resp. a trivial fibration), then by Corollary 5.16 for every object $X \in \mathcal{S}$, the induced map

$$\underline{\mathrm{Hom}}(r(X), \mathscr{F}) \to \underline{\mathrm{Hom}}(r(X), \mathscr{G})$$

is a fibration (resp. a trivial fibration). Hence the map $r^*(a)$ is a projective fibration (resp. projective trivial fibration). This implies that the pair (r^*, r_*) is a Quillen adjunction.

It remains to construct an isomorphism $r^*(X\otimes\mathbb{Q}) \simeq r(X)$ in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ functorial in X. Let $\mathscr{F}$ be an object in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. Then there are isomorphisms functorial in $\mathscr{F}$ and X

$$\begin{aligned}\mathrm{Hom}(r^*(X\otimes\mathbb{Q}), \mathscr{F}) &\simeq \mathrm{Hom}_{\mathbf{PSh}(\mathcal{S},\mathsf{Ch}(\mathbb{Q}))}(X\otimes\mathbb{Q}, r_*(\mathscr{F}))\\ &\simeq \mathrm{Hom}_{\mathsf{Ch}(\mathbb{Q})}(\mathbb{Q}, r_*(\mathscr{F})(X)) = \mathrm{Hom}_{\mathsf{Ch}(\mathbb{Q})}(\mathbb{Q}, \underline{\mathrm{Hom}}(r(X), \mathscr{F}))\\ &\simeq \mathrm{Hom}(r(X), \mathscr{F})\end{aligned}$$

(see e.g. [**3**, Proposition §4.4]). The result follows then by the Yoneda Lemma. □

5.9. Now let $\mathscr{M} \in \{\mathscr{N}, \mathscr{H}, \mathscr{P}\}$ and consider the category $\mathscr{M}(X)$. The functor $A \mapsto A(1)$ is $\mathbb{Q}$-linear exact and an auto-equivalence of category $\mathscr{M}(X)$. It induces a $\mathbb{Q}$-linear exact equivalence of categories

$$\mathsf{T}_X^{\mathscr{M}} : \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})) \to \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})).$$

For every simplicial sheaf $\mathscr{F} \in \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$, the object $\mathsf{T}_X^{\mathscr{M}}(\mathscr{F})$ is the simplicial sheaf such that for every $n \in \mathbb{N}$ and $A \in \mathscr{M}(X)$

$$\mathsf{T}_X^{\mathscr{M}}(\mathscr{F})_n(A) = \mathscr{F}_n(A(-1))[1].$$

Note that for every $A \in \mathscr{M}(X)$ we have an isomorphism (functorial in A)

$$\mathsf{cc}(\mathsf{i}(A(1)[1])) = \mathsf{T}_X^{\mathscr{M}}(\mathsf{cc}(\mathsf{i}(A))).$$

Remark 5.18. — Since the functor $\mathsf{T}_X^{\mathscr{M}}$ commutes with colimits, for every $\mathscr{F} \in \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$ and every $S \in \Delta^{\mathrm{op}}\mathsf{Sets}$ there is a canonical isomorphism

$$\mathsf{T}_X^{\mathscr{M}}(\mathscr{F}) \odot S = \mathsf{T}_X^{\mathscr{M}}(\mathscr{F} \odot S).$$

Note that $\mathsf{T}_X^{\mathscr{M}}$ is a Quillen equivalence for the canonical model structure on $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$. Let

$$\mathfrak{M}\mathscr{M}(X) := \mathsf{Sp}^{\Sigma}_{\mathsf{T}_X^{\mathscr{M}}}(\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})))$$

be the category of $\mathsf{T}_X^{\mathscr{M}}$-symmetric spectra in the category $\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$ as defined in [**3**] (see [**18**, Definition 1.1] for the non symmetric spectra). By [**3**, Lemme 4.4.35] (see also [**18**, Theorem 5.1] for a non-symmetric statement), the canonical functors

$$\mathsf{Sus}^0_{\mathsf{T}_X^{\mathscr{M}},\Sigma} : \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})) \leftrightarrows \mathfrak{M}\mathscr{M}(X) : \mathsf{Ev}_0 \tag{28}$$

are then a Quillen equivalence where the right hand side is endowed with its stable model structure.

Lemma 5.19. — *Let $\mathcal{C}$ be an essentially small category and*

$$F : \mathscr{I} \to \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$$

be a functor. Then there is a natural isomorphism

$$\mathsf{T}_X^{\mathscr{M}}(\operatorname*{hocolim}_{\mathcal{C}} F) \simeq \operatorname*{hocolim}_{\mathcal{C}}(\mathsf{T}_X^{\mathscr{M}} \circ F)$$

Proof. — By [**37**, Theorem 3.6, Proposition 4.5], the category $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ with its canonical model structure is a simplicial model category. The homotopy colimit functor is thus given by the Bousfield-Kan formula (see [**16**, Definition 19.1.2]). In other words, if G is $\mathcal{C}$-diagram , then $\operatorname{hocolim}_{\mathcal{C}} G$ is the coequalizer

$$\coprod_{\sigma:\alpha\to\alpha'} G(\alpha) \odot \mathrm{B}(\alpha' \downarrow \mathcal{C})^{\mathrm{op}} \to \coprod_{\alpha\in \mathrm{Ob}(\mathcal{C})} G(\alpha) \odot \mathrm{B}(\alpha \downarrow \mathcal{C})^{\mathrm{op}}$$

The result follows therefore from Remark 5.18. □

The Quillen equivalences (22) and (28) provides an equivalence of homotopy categories

$$\mathrm{D}(\mathbf{Sha}(\mathscr{M}(X), \mathbb{Q})) = \mathsf{Ho}(\mathbf{Sha}(\mathscr{M}(X)), \mathsf{Ch}(\mathbb{Q})) \simeq \mathsf{Ho}(\mathfrak{M}\mathscr{M}(X)) \tag{29}$$

and by Lemma 5.4 the left hand side contains $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$ as a full triangulated subcategory.

6. Perverse realization of motives

In this last section we give the construction of the realization functors. Let us briefly sketch it as a guide.

Given an affine scheme $Y \in \mathsf{SmAff}/X$, we have associated to every stratification of Y a complex of objects in $\mathscr{M}(X)$ that computes, for cellular stratifications, its $\mathscr{M}$-homology. The first step is to get rid of choices by taking an homotopy colimit over all stratifications. For functoriality, it is necessary to consider all stratifications but only the cellular ones yield the right answer (fortunately the basic Lemma shows that there are enough of them).

The realization is so far only define over SmAff/X. The next step is to extend it to all smooth quasi-projective X-schemes by a kind of homotopy left Kan extension inspired by the affine replacement functor introduced by F. Morel in [**31**, §A.2].

One then uses Proposition 5.17 to extend it further to a left Quillen functor on the category of presheaves $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ with its projective model structure. We check that it compatible with the $(\mathbf{A}^1, \text{ét})$-Bousfield localization (see Proposition 6.6). The proof boils down to checking property of the cellular complexes we beginned with). The final step is to stabilize the construction (see Proposition 6.21).

6.1. Recall that we have an exact fully faithful functor

$$\mathsf{i} : \mathsf{Ch}(\mathscr{M}(X)) \to \mathbf{Sh}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$$

and the constant simplicial functor

$$\mathsf{cc} : \mathbf{Sh}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})) \to \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})).$$

For $Y \in \mathsf{SmAff}/X$ and a stratification $Y_\bullet$ of Y, we denote by $\mathsf{ir}^{\mathscr{M}}_X(Y, Y_\bullet)$ the image of $\mathsf{r}^{\mathscr{M}}_X(Y, Y_\bullet)$ in $\mathbf{Sh}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$.

Definition 6.1. — Let $Y \in \mathsf{SmAff}/X$. We set

$$\mathsf{ra}^{\mathscr{M}}_X(Y) := \operatorname*{hocolim}_{Y_\bullet \in \mathscr{S}_Y} \mathsf{cc}(\mathsf{ir}^{\mathscr{M}}_X(Y, Y_\bullet))$$

This provides a functor

$$\mathsf{ra}^{\mathscr{M}}_X : \mathsf{SmAff}/X \to \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})).$$

Indeed let $f : Y \to Y'$ a morphism is SmAff/X. There is a functor $f_\sharp : \mathscr{S}_Y \to \mathscr{S}_{Y'}$. Hence by [**16**, Proposition 19.1.8], we have a canonical morphism

$$\operatorname*{hocolim}_{Y_\bullet \in \mathscr{S}_Y} \mathsf{cc}(\mathsf{ir}^{\mathscr{M}}_X(Y', f_\sharp(Y_\bullet))) \to \operatorname*{hocolim}_{Y'_\bullet \in \mathscr{S}_{Y'}} \mathsf{cc}(\mathsf{ir}^{\mathscr{M}}_X(Y, Y_\bullet)) =: \mathsf{ra}^{\mathscr{M}}_X(Y).$$

On the other hand we have a morphism $\mathsf{r}^{\mathscr{M}}_X(Y, -) \to \mathsf{r}^{\mathscr{M}}_X(Y', f_\sharp(-))$ of functors on $\mathscr{S}_Y$, that induces a map

$$\mathsf{ra}^{\mathscr{M}}_X(Y) := \operatorname*{hocolim}_{Y_\bullet \in \mathscr{S}_Y} \mathsf{cc}(\mathsf{ir}^{\mathscr{M}}_X(Y, Y_\bullet)) \to \operatorname*{hocolim}_{Y_\bullet \in \mathscr{S}_Y} \mathsf{cc}(\mathsf{ir}^{\mathscr{M}}_X(Y', f_\sharp(Y_\bullet))).$$

The composition provides a map $\mathsf{ra}^{\mathscr{M}}_X(Y) \to \mathsf{ra}^{\mathscr{M}}_X(Y')$ and functoriality is easy to check.

Remark 6.2. — For every stratification $Y_\bullet$ of Y, the object $\mathsf{ir}^{\mathscr{M}}_X(Y, Y_\bullet)$ is cofibrant in $\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$ by Remark 5.9. Hence, since cc is a left Quillen functor, it follows from [**16**, Theorem 18.5.2] that $\mathsf{ra}^{\mathscr{M}}_X(Y)$ is cofibrant in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$.

Let us mention the following important consequence of Lemma 4.5:

Lemma 6.3. — *Let $Y \in \mathsf{SmAff}/X$. Then the canonical morphism*

$$\operatorname*{hocolim}_{Y_\bullet \in \mathscr{A}_Y} \mathsf{cc}(\mathsf{ir}^{\mathscr{M}}_X(Y, Y_\bullet)) \to \operatorname*{hocolim}_{Y_\bullet \in \mathscr{S}_Y} \mathsf{cc}(\mathsf{ir}^{\mathscr{M}}_X(Y, Y_\bullet))$$

is a quasi-isomorphism.

Proof. — By [**16**, Theorem 19.6.7], it is enough to chek that the functor $\mathscr{A}_Y \to \mathscr{S}_Y$ is homotopy right cofinal. Let us denote by I this functor. We have to check that, for every $Y_\bullet$ in $\mathscr{S}_Y$, the nerve $\mathrm{B}(Y_\bullet \downarrow \mathsf{I})$ is contractible. This follows from Lemma 4.5 which implies that the category $Y_\bullet \downarrow \mathsf{I}$ is filtered. □

The next step is to extend the functor $\mathsf{ra}_X^{\mathscr{M}}$ to smooth quasi-projective X-schemes that may not be affine. For this we use a kind of homotopy left Kan extension inspired by the affine replacement functor introduced by F. Morel in [**31**, §A.2].

Definition 6.4. — Let $Y \in \mathsf{Sm}/X$. We set

$$\mathsf{r}_X^{\mathscr{M}}(Y) := \operatorname*{hocolim}_{(Z\to Y)\in(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}}(Z).$$

Let $I_Y : (\mathsf{SmAff}/X) \downarrow Y \to \mathsf{SmAff}/X$ be the forgetful functor defined by $I_Y(Z \to Y) = Z$. The above homotopy colimit may then be rewritten as

$$\mathsf{r}_X^{\mathscr{M}}(Y) := \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y.$$

Since $\mathsf{ra}_X^{\mathscr{M}}(Z)$ is cofibrant in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$ for every $Z \in \mathsf{SmAff}/X$, it follows from [**16**, Theorem 18.5.2] that $\mathsf{r}_X^{\mathscr{M}}(Y)$ is cofibrant in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$ as well.

Let $f : Y' \to Y$ be a morphism of smooth quasi-projective X-schemes. There is a functor

$$f_* : (\mathsf{SmAff}/X) \downarrow Y' \to (\mathsf{SmAff}/X) \downarrow Y$$

which maps a morphism $(Z \to Y')$ to the morphism $(Z \to Y)$ obtained by composition with f. Note that by definition $I_Y \circ f_* = I_{Y'}$. Hence by [**16**, Proposition 19.1.8], we have a canonical morphism

$$\mathsf{r}_X^{\mathscr{M}}(Y') := \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y'} \mathsf{ra}_X^{\mathscr{M}} \circ I_{Y'} \to \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y =: \mathsf{r}_X^{\mathscr{M}}(Y).$$

This provides a functor

$$\mathsf{r}_X^{\mathscr{M}} : \mathsf{Sm}/X \to \mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})).$$

Remark 6.5. — Denote again by $\mathsf{r}_X^{\mathscr{M}}$ the restriction of $\mathsf{r}_X^{\mathscr{M}}$ to the subcategory SmAff/X. There is a canonical morphism of functors

$$\mathsf{r}_X^{\mathscr{M}} \to \mathsf{ra}_X^{\mathscr{M}}.$$

For every $Y \in \mathsf{SmAff}/X$, the induced morphism $\mathsf{r}_X^{\mathscr{M}}(Y) \to \mathsf{ra}_X^{\mathscr{M}}(Y)$ is a weak equivalence. Indeed this follows from [**16**, Corollary 19.6.8] since $(\mathrm{Id} : Y \to Y)$ is a final object in the over category $(\mathsf{SmAff}/X) \downarrow Y$.

6.2. We may apply the construction explained in §5.8 to the functor $\mathsf{r}_X^{\mathscr{M}}$. Since $\mathsf{r}_X^{\mathscr{M}}(Y)$ is cofibrant for every $Y \in \mathsf{Sm}/X$, Proposition 5.17 yields a Quillen adjunction

$$\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}} := (\mathsf{r}_X^{\mathscr{M}})^* : \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})) \leftrightarrows \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) : (\mathsf{r}_X^{\mathscr{M}})_* =: \mathsf{RRQ}_X^{\mathscr{M},\mathrm{eff}}$$

such that the functors $\mathsf{r}_X^{\mathscr{M}}$ and $\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(- \otimes \mathbb{Q})$, are canonically isomorphic. Note that in the previous adjunction, the category of presheaves $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ is endowed with the projective model structure. To go further, we need to see that it is also compatible with the $(\mathbf{A}^1, \text{ét})$-model structure obtained by Bousfield localization.

Theorem 6.6. — *The adjunction* $(\mathsf{RLQ}_X^{\mathscr{M},\text{eff}}, \mathsf{RRQ}_X^{\mathscr{M},\text{eff}})$ *induces an adjunction*

$$\mathsf{RLQ}_X^{\mathscr{M},\text{eff}} : \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})) \leftrightarrows \Delta^{\text{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) : \mathsf{RRQ}_X^{\mathscr{M},\text{eff}}$$

where $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ *is endowed with the* $(\mathbf{A}^1, \text{ét})$*-local projective model structure.*

The proof of the Theorem relies on the universal property of Bousfield localization and Proposition 2.1. This theorem provides realization functors for effective étale motives. Let $\mathsf{RL}_X^{\mathscr{M},\text{eff}}$ be the left derived functor of $\mathsf{RLQ}_X^{\mathscr{M},\text{eff}}$ and $\mathsf{RR}_X^{\mathscr{M},\text{eff}}$ be the right derived functor of $\mathsf{RRQ}_X^{\mathscr{M},\text{eff}}$. By Theorem 6.6 we have an adjunction

$$\mathsf{RL}_X^{\mathscr{M},\text{eff}} : \mathbf{DA}^{\text{eff},\text{ét}}(X, \mathbb{Q}) \leftrightarrows \mathsf{Ho}(\Delta^{\text{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))) : \mathsf{RR}_X^{\mathscr{M},\text{eff}}.$$

Recall that we have an equivalence of triangulated categories (provided by the Quillen equivalence (22))

$$\mathrm{D}(\mathbf{Sha}(\mathscr{M}(X), \mathbb{Q})) = \mathsf{Ho}(\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))) \rightleftarrows \mathsf{Ho}(\Delta^{\text{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))).$$

Remark 6.7. — For every $Y \in \mathsf{Sm}/X$, the presheaf $Y \otimes \mathbb{Q}$ is cofibrant for the projective model structure on $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$. In particular

$$\mathsf{RL}_X^{\mathscr{M},\text{eff}}(Y \otimes \mathbb{Q}) = \mathsf{RLQ}_X^{\mathscr{M}}(Y \otimes \mathbb{Q}) \simeq \mathsf{r}_X^{\mathscr{M}}(Y).$$

6.3. In the sequel we prove the properties of the functor $\mathsf{RL}_X^{\mathscr{M}}$ needed to prove Theorem 6.6. For this it will be handy to consider the following objects:

Definition 6.8. — Let $Y \in \mathsf{SmAff}/X$. We set

$$\mathsf{TH}_X^{\mathscr{M}}(Y) := \operatorname*{colim}_{Y_\bullet \in \mathscr{A}_Y} \mathsf{iTH}_X^{\mathscr{M}}(Y, Y_\bullet).$$

This defines a complex of objects of the category $\mathbf{Sh}(\mathscr{M}(X), \mathbb{Q})$ i.e. an object in $\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$.

Remark 6.9. — By Lemma 4.5, the complex $\mathsf{TH}_X^{\mathscr{M}}(Y)$ is a also given by the colimit over all stratifications (in that case however the transition morphisms are not always quasi-isomorphisms):

$$\mathsf{TH}_X^{\mathscr{M}}(Y) := \operatorname*{colim}_{Y_\bullet \in \mathscr{S}_Y} \mathsf{iTH}_X^{\mathscr{M}}(Y, Y_\bullet).$$

In particular the term of degree $-i$ of the complexes $\mathsf{iTH}_X^{\mathscr{M}}(Y)$ is

$$\operatorname*{colim}_{Y_\bullet \in \mathscr{S}_Y} \mathsf{iTH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i) = \operatorname*{colim}_{Y_\bullet \in \mathscr{A}_Y} \mathsf{iTH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i).$$

Remark 6.10. — Note that the canonical maps

$$\begin{tikzcd} \mathsf{ra}_X^{\mathscr{M}}(Y) \arrow[r] & \mathrm{colim}_{Y_\bullet \mathscr{S}_Y} \mathsf{ccTH}_X^{\mathscr{M}}(Y)[-2d] \\ \operatorname*{hocolim}_{Y_\bullet \in \mathscr{A}_Y} \mathsf{cc}(\mathsf{ir}_X^{\mathscr{M}}(Y, Y_\bullet)) \arrow[u] \arrow[r] & \operatorname*{colim}_{Y_\bullet \in \mathscr{A}_Y} \mathsf{cc}(\mathsf{ir}_X^{\mathscr{M}}(Y, Y_\bullet)) = \mathsf{ccTH}_X^{\mathscr{M}}(Y)[-2d] \arrow[u] \end{tikzcd}$$

are weak equivalences (the vertical one on the right being even an isomorphism).

From this we obtain a functor

$$\mathsf{TH}_X^{\mathscr{M}} : \mathsf{SmAff}/X \to \mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})).$$

We prove now some elementary properties of the functor $\mathsf{TH}_X^{\mathscr{M}}$. Recall that an affine vector bundle torsor over a quasi-projective k-scheme Y is an affine scheme T and an affine morphism $T \to Y$ which is a E-torsor for some vector bundle E over Y. Recall that every quasi-projective k-scheme Y admits a an affine vector bundle torsor $T \to Y$ (see [**26**, Lemme 1.5] or [**45**, Proposition 4.3]). If Y is an affine scheme, then by [**13**, Théorème (1.3.1)] an affine vector bundle torsor $T \to Y$ is simply a vector bundle.

Lemma 6.11. — *Let $Y \in \mathsf{SmAff}/X$ and $T \to Y$ be a vector bundle. Then the canonical morphism $\mathsf{TH}_X^{\mathscr{M}}(T) \to \mathsf{TH}_X^{\mathscr{M}}(Y)$ is a quasi-isomorphism.*

Proof. — We may assume that Y is non empty. Let $Y_\bullet$ be a stratification of Y and n the smallest integer such that $Y_n = Y$. Consider the stratification $T^\bullet$ of T given by

$$T_i := \begin{cases} Y_i & \text{if } i \leqslant n \\ T & \text{if } i > n \end{cases}.$$

where Y_i is embedded in T using the zero section $Y \hookrightarrow T$. We have by Lemma 3.8

$$\mathsf{TH}_X^{\mathscr{M}}(T, Y, i) = 0$$

for all $i \in \mathbb{Z}$. Hence if $Y_\bullet$ is cellular, then $T_\bullet$ is cellular and

$$\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet) = \mathsf{TH}_X^{\mathscr{M}}(Z, Z^\bullet).$$

This implies that projection induces a quasi-isomorphism $\mathsf{TH}_X^{\mathscr{M}}(T) \to \mathsf{TH}_X^{\mathscr{M}}(Y)$ and the result follows. □

Lemma 6.12. — *Let $Y \in \mathsf{SmAff}/X$ and let $p : Y' \to Y$ be a Galois covering with Galois group G. Then the canonical morphism*

$$\mathsf{TH}_X^{\mathscr{M}}(Y')^G \to \mathsf{TH}_X^{\mathscr{M}}(Y)$$

is a quasi-isomorphism.

Proof. — We may assume that Y is non empty. Let $Y_\bullet$ be a stratification of Y. Let Y_i' be the inverse image of Y_i under p. Since p is finite étale, we have $\dim(Y_i') =$

$\dim(Y_i) \leqslant i$ and $Y'_\bullet$ is a stratification of Y' invariant under the action of G. Lemma 3.7 implies that p induces an isomorphism of complexes

$$\mathsf{TH}^{\mathscr{M}}_X(Y', Y'_\bullet)^G \to \mathsf{TH}^{\mathscr{M}}_X(Y, Y_\bullet).$$

The result follows from this. □

Remark 6.13. — If $Y' \to Y$ is an étale morphism and $Y_\bullet$ is a stratification of Y. Then $\dim(Y'_i) = \dim(Y_i)$ for every $i \in \mathbb{Z}$ where $Y'_i := Y_i \times_Y Y'$. In particular $Y'_\bullet$ is stratification of Y. We call it the induced stratification.

Let $a : Y \to X$ be an smooth affine morphism of quasi-projective k-varieties. Consider an elementary affine Nisnevich square

$$\begin{tikzcd} V \arrow[r, "u'"] \arrow[d, "e'"'] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\ U \arrow[r, "u"] & Y. \end{tikzcd}$$

Let $V_\bullet$, $U_\bullet$, $E_\bullet$ and $Y_\bullet$ be stratifications of the schemes V, U, E and Y respectively. If $U_\bullet$, $E_\bullet$ and $V_\bullet$ are induced by $Y_\bullet$, then for $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$ the long exact sequence (8) yields the short

$$\begin{tikzcd} \mathsf{TH}^{\mathscr{M}}_X(Y_i, Y_{i-1}, i+1) \arrow[r] & \mathsf{TH}^{\mathscr{M}}_X(V_i, V_{i-1}, i) \arrow[r] & \mathsf{TH}^{\mathscr{M}}_X(U_i, U_{i-1}, i) \oplus \mathsf{TH}^{\mathscr{M}}_X(E_i, E_{i-1}, i) \arrow[d] \\ & & \mathsf{TH}^{\mathscr{M}}_X(Y_i, Y_{i-1}, i) \arrow[d] \\ & & \mathsf{TH}^{\mathscr{M}}_X(V_i, V_{i-1}, i-1). \end{tikzcd}$$

For the perverse Nori motives we just have a short exact sequence (see Corollary 3.6)

$$\mathsf{TH}^{\mathscr{N}}_X(V_i, V_{i-1}, i) \to \mathsf{TH}^{\mathscr{N}}_X(U_i, U_{i-1}, i) \oplus \mathsf{TH}^{\mathscr{N}}_X(E_i, E_{i-1}, i) \to \mathsf{TH}^{\mathscr{N}}_X(Y_i, Y_{i-1}, i).$$

If the stratifications are just compatible by which we mean that $Y_\bullet$ is finer than $u_\sharp(U_\bullet)$ and $e_\sharp(E_\bullet)$, $U_\bullet$ is finer than $e'_\sharp(V_\bullet)$ and $E_\bullet$ is finer than $u'_\sharp(V_\bullet)$, then we just have morphisms

$$\mathsf{TH}^{\mathscr{M}}_X(V_i, V_{i-1}, i) \to \mathsf{TH}^{\mathscr{M}}_X(U_i, U_{i-1}, i) \oplus \mathsf{TH}^{\mathscr{M}}_X(E_i, E_{i-1}, i) \to \mathsf{TH}^{\mathscr{M}}_X(Y_i, Y_{i-1}, i).$$

This is a complex which may not be exact.

Proposition 6.14. — *Let $Y \in \mathsf{SmAff}/X$ and*

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\ U \arrow[r, "u"] & Y \end{tikzcd}$$

be an elementary affine Nisnevich square. The short sequence in $\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$

$$0 \to \mathsf{TH}^{\mathscr{M}}_X(V) \to \mathsf{TH}^{\mathscr{M}}_X(U) \oplus \mathsf{TH}^{\mathscr{M}}_X(E) \to \mathsf{TH}^{\mathscr{M}}_X(Y) \to 0$$

is exact.

Proof. — A sequence of complexes being exact if and only if it degreewise exact, it amounts to show that, for every integer $i \in \mathbb{Z}$, the sequence

$$\begin{tikzcd} 0 \arrow[d] & & \\ \operatorname*{colim}_{V_\bullet \in \mathscr{S}_V} \mathsf{iTH}_X^{\mathscr{M}}(V_i, V_{i-1}, i) \arrow[r] & \operatorname*{colim}_{U_\bullet \in \mathscr{S}_U} \mathsf{iTH}_X^{\mathscr{M}}(U_i, U_{i-1}, i) \oplus \operatorname*{colim}_{E_\bullet \in \mathscr{S}_E} \mathsf{iTH}_X^{\mathscr{M}}(E_i, E_{i-1}, i) \arrow[d] \\ & \operatorname*{colim}_{Y_\bullet \in \mathscr{S}_Y} \mathsf{iTH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i) \arrow[d] \\ & 0 \end{tikzcd} \tag{30}$$

is exact in $\mathbf{Sh}(\mathscr{M}(X), \mathbb{Q})$ (see Remark 6.9). For $W \in \{V, U, E, Y\}$, let

$$\mathscr{F}_W := \operatorname*{colim}_{W_\bullet \in \mathscr{S}_W} \mathsf{iTH}_X^{\mathscr{M}}(W_i, W_{i-1}, i)$$

where the colimit is taken in the category $\mathbf{PSha}(\mathscr{M}(X), \mathbb{Q})$ and not in the category of sheaves $\mathbf{Sha}(\mathscr{M}(X), \mathbb{Q})$. For every $A \in \mathscr{M}(X)$, one has

$$\begin{aligned} \mathscr{F}_W(A) &= \operatorname*{colim}_{W_\bullet \in \mathscr{S}_W} \Gamma(A, \mathsf{iTH}_X^{\mathscr{M}}(W_i, W_{i-1}, i)) \\ &= \operatorname*{colim}_{W_\bullet \in \mathscr{S}_W} \operatorname{Hom}_{\mathscr{M}(X)}(A, \mathsf{TH}_X^{\mathscr{M}}(W_i, W_{i-1}, i)). \end{aligned}$$

Note that the sequence (30) is the induced sequence

$$0 \to a_{\mathrm{epi}}\mathscr{F}_V \to a_{\mathrm{epi}}\mathscr{F}_U \oplus a_{\mathrm{epi}}\mathscr{F}_E \to a_{\mathrm{epi}}\mathscr{F}_Y \to 0$$

so we may use Remark 5.3 to show its exactness. Let us prove the exactness on the right and on the left (the exactness at the center is proved similarly using Lemma 3.5 or Corollary 3.6).

Let $A \in \mathscr{M}(X)$ and $\alpha \in \mathscr{F}_Y(A)$. There exists a stratification $Y_\bullet$ of Y and an element $\alpha_{Y_\bullet} \in \operatorname{Hom}_{\mathscr{M}(X)}(A, \mathsf{TH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i))$ that lifts α. Let $U_\bullet$, $E_\bullet$ and $V_\bullet$ the induced stratifications. Let $V'_\bullet$ be a cellular stratification of V finer than $V_\bullet$, $Y''_\bullet$ be a stratification of Y such that $h(V'_i) \subseteq Y''_i$ for every $i \in \mathbb{Z}$. Let $E''_\bullet$, $U''_\bullet$ and $V''_\bullet$ the stratifications induced by $Y''_\bullet$. Let us show that the morphism

$$\mathsf{TH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i) \to \mathsf{TH}_X^{\mathscr{M}}(Y''_i, Y''_{i-1}, i)$$

factorizes through the image of the morphism

$$\mathsf{TH}_X^{\mathscr{M}}(U''_i, U''_{i-1}, i) \oplus \mathsf{TH}_X^{\mathscr{M}}(E''_i, E''_{i-1}, i) \to \mathsf{TH}_X^{\mathscr{M}}(Y''_i, Y''_{i-1}, i).$$

Using the exact and faithful functor $\mathscr{N}(X) \to \mathscr{P}(X)$, we may assume that $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. In that case, there is by Lemma 3.5 a commutative diagram in $\mathscr{M}(X)$

$$\begin{tikzcd}
\mathsf{TH}^{\mathscr{M}}_X(U''_i, U''_{i-1}, i) \oplus \mathsf{TH}^{\mathscr{M}}_X(E''_i, E''_{i-1}, i) \arrow[r] & \mathsf{TH}^{\mathscr{M}}_X(Y''_i, Y''_{i-1}, i) \arrow[r] & \mathsf{TH}^{\mathscr{M}_X}(V''_i, V''_{i-1}, i-1) \\
 & & \mathsf{TH}^{\mathscr{M}}_X(V'_i, V'_{i-1}, i-1) \arrow[u] \\
\mathsf{TH}^{\mathscr{M}}_X(U_i, U_{i-1}, i) \oplus \mathsf{TH}^{\mathscr{M}}_X(E_i, E_{i-1}, i) \arrow[r] \arrow[uu] & \mathsf{TH}^{\mathscr{M}}_X(Y_i, Y_{i-1}, i) \arrow[r] \arrow[uu] & \mathsf{TH}^{\mathscr{M}}_X(V_i, V_{i-1}, i-1) \arrow[u]
\end{tikzcd}$$

with exact rows. The result then follows from the fact that $\mathsf{TH}^{\mathscr{M}}_X(V'_i, V'_{i-1}, i-1) = 0$ since $V'_\bullet$ is a cellular stratification. This implies the existence of an epimorphism $B \twoheadrightarrow A$ in $\mathscr{M}(X)$ and elements

$$(\beta_{U''_\bullet}, \gamma_{E''_\bullet}) \in \mathrm{Hom}_{\mathscr{M}(X)}(B, \mathsf{TH}^{\mathscr{M}}_X(U''_i, U''_{i-1}, i)) \oplus \mathrm{Hom}_{\mathscr{M}(X)}(B, \mathsf{TH}^{\mathscr{M}}_X(E''_i, E''_{i-1}, i)$$

such that the image of $(\beta_{U''_\bullet}, \gamma_{E''_\bullet})$ in

$$\Gamma(B, \mathsf{i}\mathsf{TH}^{\mathscr{M}}_X(Y''_i, Y''_{i-1}, i)) = \mathrm{Hom}_{\mathscr{M}(X)}(B, \mathsf{TH}^{\mathscr{M}}_X(Y''_i, Y''_{i-1}, i))$$

is equal to the image of $\alpha_{Y_\bullet}$. Let (β, γ) the image of $(\beta_{U''_\bullet}, \gamma_{E''_\bullet})$ in $\mathscr{F}_U(B) \oplus \mathscr{F}_E(B)$. Then the image of (β, γ) in $\mathscr{F}_Y(B)$ is equal to the image of α. This shows the exactness on the right.

Let $A \in \mathscr{M}(X)$ and $\alpha \in \mathscr{F}_V(A)$ such that $\alpha = 0$ in $\mathscr{F}_U(A) \oplus \mathscr{F}_E(A)$. Let $V_\bullet$ be a stratification of V and $\alpha_{V_\bullet}$ an element in $\mathrm{Hom}_{\mathscr{M}(X)}(A, \mathsf{TH}^{\mathscr{M}}_X(V_i, V_{i-1}, i))$ that lifts α. There exist a stratification $U_\bullet$ of U and a stratification $E_\bullet$ of E, both compatible with $V_\bullet$, such that $\alpha_{V_\bullet} = 0$ in

$$\mathrm{Hom}_{\mathscr{M}(X)}(A, \mathsf{TH}^{\mathscr{M}}_X(U_i, U_{i-1}, i)) \oplus \mathrm{Hom}_{\mathscr{M}(X)}(A, \mathsf{TH}^{\mathscr{M}}_X(E_i, E_{i-1}, i)).$$

Let $Y_\bullet$ be a stratification of Y compatible with $U_\bullet$ and $V_\bullet$. Let $Y'_\bullet$ be a cellular stratification finer than $Y_\bullet$ and let $V'_\bullet$, $U'_\bullet$ and $E'_\bullet$ be the induced stratifications. The morphism

$$\mathsf{TH}^{\mathscr{M}}_X(V'_i, V'_{i-1}, i) \to \mathsf{TH}^{\mathscr{M}}_X(U'_i, U'_{i-1}, i) \oplus \mathsf{TH}^{\mathscr{M}}_X(E'_i, E'_{i-1}, i)$$

is a monomorphism. Indeed using the faithful exact functor $\mathscr{N}(X) \to \mathscr{P}(X)$, we may assume that $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. In that case, by Lemma 3.5, one has the commutative diagram in which the top row is exact

$$\begin{tikzcd}
\mathsf{TH}^{\mathscr{M}}_X(Y'_i, Y'_{i-1}, i+1) \arrow[r] & \mathsf{TH}^{\mathscr{M}}_X(V'_i, V'_{i-1}, i) \arrow[r] & \mathsf{TH}^{\mathscr{M}}_X(U'_i, U'_{i-1}, i) \oplus \mathsf{TH}^{\mathscr{M}}_X(E'_i, E'_{i-1}, i) \\
 & \mathsf{TH}^{\mathscr{M}}_X(V_i, V_{i-1}, i) \arrow[r] \arrow[u] & \mathsf{TH}^{\mathscr{M}}_X(U_i, U_{i-1}, i) \oplus \mathsf{TH}^{\mathscr{M}}_X(E_i, E_{i-1}, i). \arrow[u]
\end{tikzcd}$$

Since $Y'_\bullet$ is cellular, $\mathsf{TH}^{\mathscr{M}}_X(Y'_i, Y'_{i-1}, i+1) = 0$ and the claim follows. Hence the image of $\alpha_{V_\bullet}$ in $\mathrm{Hom}_{\mathscr{M}(X)}(A, \mathsf{TH}^{\mathscr{M}}_X(V'_i, V'_{i-1}, i)$ vanishes and therefore $\alpha = 0$ in $\mathscr{F}_V(A)$. This shows the exactness on the left. $\square$

6.4. Let us state two consequences of the previous results:

Corollary 6.15. — *Let $Y \in \mathsf{SmAff}/X$ and $T \to Y$ be a vector bundle. Then the canonical morphism*

$$\mathsf{ra}_X^{\mathscr{M}}(T) \to \mathsf{ra}_X^{\mathscr{M}}(Y)$$

is a weak equivalence in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$.

Proof. — This follows from Remark 6.10 and Lemma 6.11 □

Corollary 6.16. — *Let*

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\ U \arrow[r, "u"] & Y \end{tikzcd}$$

be an elementary affine Nisnevich square. Then the following square is homotopy cocartesian in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$

$$\begin{tikzcd} \mathsf{ra}_X^{\mathscr{M}}(V) \arrow[r] \arrow[d] & \mathsf{ra}_X^{\mathscr{M}}(E) \arrow[d] \\ \mathsf{ra}_X^{\mathscr{M}}(U) \arrow[r] & \mathsf{ra}_X^{\mathscr{M}}(Y). \end{tikzcd}$$

Proof. — This follows immediately from Proposition 6.14 using a classical result of homological algebra (see e.g. [**27**, Proposition 1.7.5]) and Remark 6.10. □

Proposition 6.17. — *Let $Y \in \mathsf{Sm}/X$ and $T \to Y$ be an affine vector bundle torsor. Then the morphism*

$$\mathsf{r}_X^{\mathscr{M}}(T) \to \mathsf{r}_X^{\mathscr{M}}(Y)$$

is a weak equivalence in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$.

Proof. — The proof of the proposition follows the line of [**46**, Proposition 3.11]. Let $p : T \to Y$ be an affine vector bundle torsor. We have to show that the morphism

$$\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow T} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \circ p_* \to \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y$$

induced by the functor p_* is a weak equivalence ($I_Y \circ p_* = I_T$). Consider the functor obtained by base change along p

$$\begin{aligned} p^* : (\mathsf{SmAff}/X) \downarrow Y &\to (\mathsf{SmAff}/X) \downarrow T \\ (Z \to Y) &\mapsto (T \times_Y Z \to T). \end{aligned}$$

Note that this functor is well-defined. Indeed $T \times_Y Z \to Z$ is an affine vector bundle torsor over an affine scheme Z and therefore $T \times_Y Z$ is also an affine scheme. As shown in [**46**, Proof of Proposition 3.11], the functor p^* is homotopy right cofinal and the canonical morphism

$$\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \to \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow T} \mathsf{ra}_X^{\mathscr{M}} \circ I_T. \tag{31}$$

is therefore a weak equivalence by [**16**, Theorem 19.6.7]. On the other hand, the morphisms of affine schemes $(I_T \circ p^*)(Z \to Y) = T \times_Y Z \to Z = I_Y(Z \to Y)$ define a morphism of functors $I_T \circ p^* \to I_Y$ and thus yield morphisms of functors

$$\mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \to \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \qquad \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \circ p_* \to \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \circ p_*. \tag{32}$$

Since $p_Z : T \times_Y Z \to Z$ is an affine vector bundle torsor, by Lemma 6.15, the morphisms (32) are weak equivalences of diagrams and therefore, by [**16**, Theorem 19.4.2], the maps

$$\underset{(\mathsf{SmAff}/X)\downarrow Y}{\mathrm{hocolim}} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \to \underset{(\mathsf{SmAff}/X)\downarrow Y}{\mathrm{hocolim}} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \tag{33}$$

$$\underset{(\mathsf{SmAff}/X)\downarrow Y}{\mathrm{hocolim}} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \circ p_* \to \underset{(\mathsf{SmAff}/X)\downarrow Y}{\mathrm{hocolim}} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \circ p_* \tag{34}$$

are weak equivalences. We have a commutative square

$$\begin{tikzcd} \underset{(\mathsf{SmAff}/X)\downarrow T}{\mathrm{hocolim}} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \circ p_* \arrow[r] \arrow[d, "(34)"] & \underset{(\mathsf{SmAff}/X)\downarrow Y}{\mathrm{hocolim}} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \arrow[d, "(33)"] \\ \underset{(\mathsf{SmAff}/X)\downarrow T}{\mathrm{hocolim}} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \circ p_* \arrow[r] & \underset{(\mathsf{SmAff}/X)\downarrow Y}{\mathrm{hocolim}} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y. \end{tikzcd}$$

Since (33) and (33) are weak equivalences, it is enough to show that the top horizontal map is a weak equivalence. The composition

$$\underset{(\mathsf{SmAff}/X)\downarrow T}{\mathrm{hocolim}} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \circ p_* \to \underset{(\mathsf{SmAff}/X)\downarrow Y}{\mathrm{hocolim}} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \xrightarrow{(31)} \underset{(\mathsf{SmAff}/X)\downarrow T}{\mathrm{hocolim}} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \tag{35}$$

of this map with (31) is the canonical map induced by the functor

$$p^* \circ p_* : (\mathsf{SmAff}/X) \downarrow T \to (\mathsf{SmAff}/X) \downarrow T.$$

Since this functor is homotopy right cofinal (see [**46**, Proof of Proposition 3.11]), the composition (35) is a weak equivalence by [**16**, Theorem 19.6.7]. This concludes the proof since (31) is a weak equivalence. □

Proposition 6.18. — *Let $Y \in \mathsf{Sm}/X$.*

1. *Let U, E be an open cover of Y. Then the square*

$$\begin{tikzcd} \mathsf{r}_X^{\mathscr{M}}(V) \arrow[r] \arrow[d] & \mathsf{r}_X^{\mathscr{M}}(E) \arrow[d] \\ \mathsf{r}_X^{\mathscr{M}}(U) \arrow[r] & \mathsf{r}_X^{\mathscr{M}}(Y) \end{tikzcd}$$

   is homotopy cocartesian in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$.
2. *The morphism*

$$\mathsf{r}_X^{\mathscr{M}}(Y \times_k \mathbf{A}^1_k) \to \mathsf{r}_X^{\mathscr{M}}(Y)$$

   est un weak equivalence.

Proof. — Proposition 6.17 allows the use of Jouanolou's trick. The proof of the first statement is then completely similar to the proof of the Mayer-Vietoris property for homotopy invariant K-theory given in [**45**, Theorem 5.1]. The details are left to the readers. Let us proof the second statement. Let $T \to Y$ be an affine vector bundle torsor (since Y is quasi-projective over k, the existence of such a torsor as been shown in [**26**, Lemme 1.5]). We have commutative squares

$$\begin{tikzcd} \mathsf{ra}^{\mathscr{M}}_X(T \times_k \mathbf{A}^1_k) \arrow[d] & \mathsf{r}^{\mathscr{M}}_X(T \times_k \mathbf{A}^1_k) \arrow[l] \arrow[r] \arrow[d] & \mathsf{r}^{\mathscr{M}}_X(Y \times_k \mathbf{A}^1_k) \arrow[d] \\ \mathsf{ra}^{\mathscr{M}}_X(T) & \mathsf{r}^{\mathscr{M}}_X(T) \arrow[l] \arrow[r] & \mathsf{r}^{\mathscr{M}}_X(Y). \end{tikzcd}$$

By Proposition 6.17 and Remark 6.5 the horizontal morphisms are weak equivalences. The result follows then from Lemma 6.15 which ensures that the vertical arrow on the left is a weak equivalence. □

Remark 6.19. — Let $Y \in \mathsf{Sm}/X$ and $\mathscr{U} := \{U_i \hookrightarrow Y\}_{i \in I}$ be a finite open cover of Y. Let U be the disjoint union of the U_i's. We have the usual Čech simplicial object $\check{\mathrm{C}}(\mathscr{U}) : \Delta^{\mathrm{op}} \to \mathsf{SmAff}/X$ such that for every $n \in \Delta$, $\check{\mathrm{C}}(\mathscr{U})_n$ is the fiber product over X of n copies of U:

$$\check{\mathrm{C}}(\mathscr{U})_n = U \times_X \cdots \times_X U.$$

One can show by induction on the number of open subsets in $\mathscr{U}$ (see [**45**, Theorem 6.3]) that the canonical morphism

$$\mathsf{r}^{\mathscr{M}}_X(Y, \mathscr{U}) := \operatorname*{hocolim}_{\Delta} \mathsf{r}^{\mathscr{M}}_X(\check{\mathrm{C}}(\mathscr{U})) \to \mathsf{r}^{\mathscr{M}}_X(Y)$$

is a weak equivalence.

Lemma 6.20. — *Let $Y \in \mathsf{SmAff}/X$ and let $p : Y' \to Y$ be a Galois covering with Galois group G. Then the canonical morphism*

$$\mathsf{r}^{\mathscr{M}}_X(Y')^G \to \mathsf{r}^{\mathscr{M}}_X(Y)$$

is a weak equivalence.

Proof. — Let $T \to Y$ be an affine vector bundle torsor and $p : T' \to T$ the Galois cover obtained by base change. We have commutative squares

$$\begin{tikzcd} \mathsf{ra}^{\mathscr{M}}_X(T')^G \arrow[d] & \mathsf{r}^{\mathscr{M}}_X(T')^G \arrow[l] \arrow[r] \arrow[d] & \mathsf{r}^{\mathscr{M}}_X(Y')^G \arrow[d] \\ \mathsf{ra}^{\mathscr{M}}_X(T) & \mathsf{r}^{\mathscr{M}}_X(T) \arrow[l] \arrow[r] & \mathsf{r}^{\mathscr{M}}_X(Y). \end{tikzcd}$$

By Proposition 6.17 and Remark 6.5 the horizontal morphisms are weak equivalences. The result follows then from Remark 6.10 and Lemma 6.12 which ensure that the vertical arrow on the left is a weak equivalence. □

Proof of Theorem 6.6. — Let us first remark that $\mathsf{RL}_X^{\mathscr{M},\mathrm{eff}}(\emptyset\otimes\mathbb{Q})=0$. Let $Y\in\mathsf{Sm}/X$ and

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\ U \arrow[r, "u"] & Y \end{tikzcd}$$

be either a Zariski square or an affine Nisnevich square. By Proposition 6.18 and Corollary 6.16, the square

$$\begin{tikzcd} \mathsf{RL}_X^{\mathscr{M},\mathrm{eff}}(V\otimes\mathbb{Q}) \arrow[r] \arrow[d] & \mathsf{RL}_X^{\mathscr{M},\mathrm{eff}}(E\otimes\mathbb{Q}) \arrow[d] \\ \mathsf{RL}_X^{\mathscr{M},\mathrm{eff}}(U\otimes\mathbb{Q}) \arrow[r] & \mathsf{RL}_X^{\mathscr{M},\mathrm{eff}}(Y\otimes\mathbb{Q}) \end{tikzcd}$$

is cocartesian in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q}))$ (here we have also used Remarks 6.5 and 6.7). One the other hand by Proposition 6.18 and Remark 6.7, the morphism

$$\mathsf{RL}_X^{\mathscr{M},\mathrm{eff}}(\mathbf{A}^1_Y\otimes\mathbb{Q})\to\mathsf{RL}_X^{\mathscr{M},\mathrm{eff}}(Y\otimes\mathbb{Q})$$

is an isomorphism in $\mathsf{Ho}(\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q})))$. If $p:Y'\to Y$ is a Galois covering with Galois group G, then by Remark 6.7 and Lemma 6.20 the morphism

$$\mathsf{RL}_X^{\mathscr{M},\mathrm{eff}}((Y'\otimes\mathbb{Q})^G)\to\mathsf{RL}_X^{\mathscr{M},\mathrm{eff}}(Y\otimes\mathbb{Q})$$

is an isomorphism in $\mathsf{Ho}(\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q})))$. It follows that $\mathsf{RL}_X^{\mathscr{M},\mathrm{eff}}$ sends the morphisms in (2) to isomorphisms in the homotopy category and Theorem 6.6 follows from the universal property of Bousfield localizations. □

6.5. It remains to stabilize the above construction in order to obtain a realization functor also for motives that may not be effective. The key result that we need is the following proposition.

Proposition 6.21. — *There exists a natural transformation*

$$\begin{tikzcd} \mathbf{PSh}(\mathsf{Sm}/X,\mathsf{Ch}(\mathbb{Q})) \arrow[r, "\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}"] \arrow[d, "T_X\otimes -"'] & \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q})) \arrow[d, "\mathsf{T}_X^{\mathscr{M}}"] \arrow[dl, phantom, "\Downarrow\rho"] \\ \mathbf{PSh}(\mathsf{Sm}/X,\mathsf{Ch}(\mathbb{Q})) \arrow[r, "\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}"] & \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q})) \end{tikzcd}$$

such that

$$\rho_{\mathscr{X}}:(\mathsf{T}_X^{\mathscr{M}}\circ\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}})(\mathscr{X})\to\mathsf{RQ}_X^{\mathscr{M}}(T_X\otimes\mathscr{X})$$

is a weak equivalence for every presheaf $\mathscr{X}\in\mathbf{PSh}(\mathsf{Sm}/X,\mathsf{Ch}(\mathbb{Q}))$.

Using the natural transformation ρ, we can construct a functor

$$\mathsf{RLQ}_X^{\mathscr{M}}:\mathsf{Sp}^{\Sigma}_{T_X}(\mathbf{PSh}(\mathsf{Sm}/X),\mathsf{Ch}(\mathbb{Q}))\to\mathfrak{M}\mathscr{M}(X):=\mathsf{Sp}^{\Sigma}_{\mathsf{T}_X^{\mathscr{M}}}(\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q}))).$$

Given a symmetric T_X-spectra $\mathscr{X} := (\mathscr{X}_n)_{n\in\mathbb{N}}$ the image $\mathsf{RLQ}_X^{\mathscr{M}}(\mathscr{X})$ of $\mathscr{X}$ is the symmetric $\mathsf{T}_X^{\mathscr{M}}$-spectra with $\mathsf{RLQ}_X^{\mathscr{M}}(\mathscr{X})_n := \mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(\mathscr{X}_n)$ and assembly maps given by the composition

$$\mathsf{T}_X^{\mathscr{M}}\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(\mathscr{X}_n) \xrightarrow{\rho_{\mathscr{X}}} \mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(T_X\otimes\mathscr{X}_n) \to \mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(\mathscr{X}_{n+1})$$

By [**3**, Lemme 4.3.34], the functor $\mathsf{RLQ}_X^{\mathscr{M}}$ is a left Quillen functor with respect to the $(\mathbf{A}^1,\text{ét})$-local stable projective model structure on the left hand side and the stable model structure on the right hand side. One obtains a Quillen adjunction

$$\mathsf{RLQ}_X^{\mathscr{M}} : \mathsf{Sp}_{T_X}^{\Sigma}(\mathbf{PSh}(\mathsf{Sm}/X),\mathsf{Ch}(\mathbb{Q})) \leftrightarrows \mathfrak{M}\mathscr{M}(X) : \mathsf{RRQ}_X^{\mathscr{M}}.$$

Via the Quillen equivalences (22) and (28), and one may view the above adjunction as a Quillen adjunction

$$\mathsf{Sp}_{T_X}^{\Sigma}(\mathbf{PSh}(\mathsf{Sm}/X),\mathsf{Ch}(\mathbb{Q})) \leftrightarrows \mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q})).$$

Taking the Quillen derived functors, and using the equivalences (29), one gets an adjunction on the homotopy categories

$$\mathsf{RL}_X^{\mathscr{M}} : \mathbf{DA}^{\text{ét}}(X,\mathbb{Q}) \rightleftarrows \mathrm{D}(\mathbf{Sha}(\mathscr{M}(X),\mathbb{Q})) : \mathsf{RR}_X^{\mathscr{M}}.$$

Recall that the full triangulated category $\mathbf{DA}_{\mathsf{ct}}^{\text{ét}}(X,\mathbb{Q})$ of constructible motives is defined as the smallest triangulated subcategory of $\mathbf{DA}^{\text{ét}}(X,\mathbb{Q})$ stable by direct factors and containing the homological motives of smooth quasi-projective X-schemes (or equivalently smooth affine X-schemes by Mayer-Vietoris). Since by construction for every affine smooth X-scheme Y, the image lands in the full triangulated category $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$ of $\mathrm{D}(\mathbf{Sha}(\mathscr{M}(X),\mathbb{Q}))$, the above functor induces a triangulated functor

$$\mathbf{DA}_{\mathsf{ct}}^{\text{ét}}(X,\mathbb{Q}) \to \mathrm{D}^{\mathrm{b}}(\mathscr{M}(X)).$$

6.6. It remains to prove Proposition 6.21. The proof is slightly technical, as we have to unwind the construction of the functor $\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}$ to construct step by step the natural transformation ρ. It essentially boils down to properties of cellular complexes associated with specific stratifications. Namely we have the following lemma:

Lemma 6.22. — *Let $Y\in\mathsf{SmAff}/X$. There exists a morphism*

$$\mathsf{T}_X^{\mathscr{M}}(\mathsf{ra}_X^{\mathscr{M}}(Y)) \to \mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})$$

in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q}))$ such that the induced morphism

$$\mathsf{ra}_X^{\mathscr{M}}(Y)\oplus\mathsf{T}_X^{\mathscr{M}}(\mathsf{ra}_X^{\mathscr{M}}(Y)) \to \mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})$$

is a weak equivalence (here the morphism $\mathsf{ra}_X^{\mathscr{M}}(Y)\to\mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})$ is the morphism induced by the unit section of $\mathbf{G}_{m,Y}$).

Proof. — Let $Y_\bullet$ be a stratification of Y. Consider the stratication $\mathsf{G}(Y_\bullet)$ of the quasi-projective k-scheme $\mathbf{G}_{m,Y}$ defined by the closed subsets $\mathsf{G}(Y_\bullet)_i := Y_{i-1}\times_k\mathbf{G}_{m,k}$. By Lemma 3.9, the complex $\mathsf{TH}_X^{\mathscr{M}}(Y,Y_\bullet)(1)[1]$ is a direct summand of the complex $\mathsf{TH}_X^{\mathscr{M}}(\mathbf{G}_{m,Y},\mathsf{G}(Y_\bullet))$. The inclusion as a direct factors induces a morphism of functors on $\mathscr{S}_Y$

$$\mathsf{TH}_X^{\mathscr{M}}(Y,-)(1)[1] \to \mathsf{TH}_X^{\mathscr{M}}(\mathbf{G}_{m,Y},\mathsf{G}(-))$$

and thus a morphism of functors

$$\mathsf{T}_X^{\mathscr{M}}(\mathsf{cc}(\mathsf{iTH}_X^{\mathscr{M}}(Y,-))) \to \mathsf{cc}(\mathsf{iTH}_X^{\mathscr{M}}(\mathbf{G}_{m,Y},\mathsf{G}(-))).$$

Taking homotopy colimits, we obtain a morphism in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q}))$

$$\operatorname*{hocolim}_{Y_\bullet\in\mathscr{S}_Y}\mathsf{T}_X^{\mathscr{M}}(\mathsf{cc}(\mathsf{iTH}_X^{\mathscr{M}}(Y,-))) \to \operatorname*{hocolim}_{Y_\bullet\in\mathscr{S}_Y}\mathsf{cc}(\mathsf{iTH}_X^{\mathscr{M}}(\mathbf{G}_{m,Y},\mathsf{G}(-))) \to \mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})$$

where the second morphism is the canonical morphism associated with the functor $\mathsf{G}:\mathscr{S}_Y\to\mathscr{S}_{\mathbf{G}_{m,Y}}$ (see [**16**, Proposition 19.1.8]).

By Lemma 5.19, there is a canonical isomorphism

$$\mathsf{T}_X^{\mathscr{M}}(\mathsf{ra}_X^{\mathscr{M}}(Y)) := \mathsf{T}_X^{\mathscr{M}}(\operatorname*{hocolim}_{Y_\bullet\in\mathscr{S}_Y}\mathsf{cc}(\mathsf{iTH}_X^{\mathscr{M}}(Y,-))) \simeq \operatorname*{hocolim}_{Y_\bullet\in\mathscr{S}_Y}\mathsf{T}_X^{\mathscr{M}}(\mathsf{cc}(\mathsf{iTH}_X^{\mathscr{M}}(Y,-)))$$

This provides the desired morphism. □

Remark 6.23. — The morphisms constructed in the proof of Lemma 6.22 are functorial in Y and define a morphism of functors

$$\mathsf{T}_X^{\mathscr{M}}\circ\mathsf{ra}_X^{\mathscr{M}} \to \mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,-})$$

on SmAff/X.

To prove Proposition 6.21 we will also need the following lemma.

Lemma 6.24. — *Let $\mathsf{c}_X^{\mathscr{M}}$ be the cokernel of the natural transformation $\mathsf{r}_X^{\mathscr{M}}\to\mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,-})$ given by the unit section. Then there is an isomorphism of functors*

$$(\mathsf{c}_X^{\mathscr{M}})^* = (\mathsf{r}_X^{\mathscr{M}})^*(T_X\otimes -).$$

Proof. — By definition $\mathsf{c}_X^{\mathscr{M}}$ is a functor $\mathsf{Sm}/X\to\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q}))$ and for every smooth quasi-projective X-scheme Y, one has a short exact sequence

$$0\to\mathsf{r}_X^{\mathscr{M}}(Y)\to\mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})\to\mathsf{c}_X^{\mathscr{M}}(Y)\to 0. \tag{36}$$

The endofunctor $(\mathbf{G}_{m,X}\otimes\mathbb{Q})\otimes -$ of the category $\mathbf{PSh}(\mathsf{Sm}/X,\mathsf{Ch}(\mathbb{Q}))$ admits $\mathscr{H}om(\mathbf{G}_{m,X}\otimes\mathbb{Q},-)$ has right adjoint (here $\mathscr{H}om$ denotes the internal Hom in the category of presheaves on Sm/X). For $\mathscr{X}\in\mathbf{PSh}(\mathsf{Sm}/X,\mathsf{Ch}(\mathbb{Q}))$, the presheaf $\mathscr{H}om(\mathbf{G}_{m,X}\otimes\mathbb{Q},\mathscr{X})$ being nothing but the presheaf $Y\mapsto\mathscr{X}(\mathbf{G}_{m,Y})$. It follows that the functor $(\mathsf{r}_X^{\mathscr{M}})^*((\mathbf{G}_{m,X}\otimes\mathbb{Q})\otimes -)$ is left adjoint to the functor $\mathscr{F}\mapsto\underline{\mathrm{Hom}}(\mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,-}),\mathscr{F})$ and is therefore isomorphic to the functor $(\mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,-}))^*$. For every $\mathscr{X}\in\mathbf{PSh}(\mathsf{Sm}/X,\mathsf{Ch}(\mathbb{Q}))$, this isomorphism fits into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathsf{r}_X^{\mathscr{M}})^*((X\otimes\mathbb{Q})\otimes\mathscr{X}) \arrow[r] \arrow[d, dash, "\text{iso.}"'] & (\mathsf{r}_X^{\mathscr{M}})^*((\mathbf{G}_{m,X}\otimes\mathbb{Q})\otimes\mathscr{X}) \arrow[r] \arrow[d, dash, "\text{iso.}"'] & (\mathsf{r}_X^{\mathscr{M}})^*(T_X\otimes\mathscr{X}) \arrow[r] & 0 \\
& (\mathsf{r}_X^{\mathscr{M}})^*(\mathscr{X}) \arrow[r] & (\mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,-}))^*(\mathscr{X}) \arrow[r] & (\mathsf{c}_X^{\mathscr{M}})^*(\mathscr{X}) \arrow[r] & 0
\end{tikzcd}$$

The rows in this diagram are exact. For the upper row this follows from the fact that $(r_X^{\mathscr{M}})^*$ is right exact (it is a left adjoint). For the lower row this follows from the short exact sequences (36) and the definition of (27) as a colimit. This provides an isomorphism of functors

$$(\mathsf{c}_X^{\mathscr{M}})^* = (\mathsf{r}_X^{\mathscr{M}})^*(T_X \otimes -)$$

as desired. □

Proof of Proposition 6.21. — By construction

$$\mathsf{T}_X^{\mathscr{M}}(\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(\mathscr{X})) = \mathsf{T}_X^{\mathscr{M}}((\mathsf{r}_X^{\mathscr{M}})^*(\mathscr{X})) = (\mathsf{T}_X^{\mathscr{M}} \circ \mathsf{r}_X^{\mathscr{M}})^*(\mathscr{X})$$

and

$$\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(T_X \otimes \mathscr{X}) = (\mathsf{r}_X^{\mathscr{M}})^*(T_X \otimes \mathscr{X})$$

hence it is enough to construct a natural transformation

$$\vartheta : (\mathsf{T}_X^{\mathscr{M}} \circ \mathsf{r}_X^{\mathscr{M}})^* \to (\mathsf{r}_X^{\mathscr{M}})^*(T_X \otimes -)$$

such that $\vartheta_{\mathscr{X}}$ is a weak equivalence for every $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$. By Lemma 6.24, it is therefore enough to construct a natural transformation

$$\varrho : \mathsf{T}_X^{\mathscr{M}} \circ \mathsf{r}_X^{\mathscr{M}} \to \mathsf{c}_X^{\mathscr{M}}$$

such that ϱ_Y is a weak equivalence for every $Y \in \mathsf{Sm}/X$.

Let us first extend Lemma 6.22 to smooth quasi-projective X-schemes which may not be affine. For $Y \in \mathsf{Sm}/X$, we construct a morphism

$$\mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) \to \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})$$

as follows. Consider the functor

$$\begin{aligned} \mathbf{G}_m : (\mathsf{SmAff}/X) \downarrow Y &\to (\mathsf{SmAff}/X) \downarrow \mathbf{G}_{m,Y} \\ (Z \to Y) &\mapsto (\mathbf{G}_{m,Z} \to \mathbf{G}_{m,Y}). \end{aligned}$$

and the induced morphism (see [**16**, Proposition 19.1.8])

$$\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_{\mathbf{G}_{m,Y}} \circ \mathbf{G}_m \to \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow \mathbf{G}_{m,Y}} \mathsf{ra}_X^{\mathscr{M}} \circ I_{\mathbf{G}_{m,Y}} =: \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}).$$

Note that $I_{\mathbf{G}_{m,Y}} \circ \mathbf{G}_m = \mathbf{G}_{m,-} \circ I_Y$. By Remark 6.23, the morphisms of Lemma 6.22 induce thus a morphism of functors

$$\mathsf{T}_X^{\mathscr{M}} \circ \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \to \mathsf{ra}_X^{\mathscr{M}} \circ I_{\mathbf{G}_{m,Y}} \circ \mathbf{G}_m$$

This provides a morphism

$$\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{T}_X^{\mathscr{M}} \circ \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \to \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow \mathbf{G}_{m,Y}} \mathsf{ra}_X^{\mathscr{M}} \circ I_{\mathbf{G}_{m,Y}} =: \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}).$$

By Lemma 5.19, there is a canonical isomorphism

$$\mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) := \mathsf{T}_X^{\mathscr{M}}\left(\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y\right) = \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{T}_X^{\mathscr{M}} \circ \mathsf{ra}_X^{\mathscr{M}} \circ I_Y.$$

Note that for every affine scheme $Y \in \mathsf{SmAff}/X$ the square

$$\begin{tikzcd} \mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) \arrow[r] \arrow[d] & \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}) \arrow[d] \\ \mathsf{T}_X^{\mathscr{M}}(\mathsf{ra}_X^{\mathscr{M}}(Y)) \arrow[r] & \mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}) \end{tikzcd}$$

is commutative where the vertical morphisms are the weak equivalences of Remark 6.5 and the lower horizontal morphism is the morphism contructed in Lemma 6.22.

It follows from Lemma 6.22 and Jouanolou's trick, that the induced morphism

$$\mathsf{r}_X^{\mathscr{M}}(Y) \oplus \mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) \to \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})$$

(given by the unit section of the first summand) is a weak equivalence. Indeed Let $T \to Y$ be an affine vector bundle torsor. We have then a commutative diagram

$$\begin{tikzcd}
\mathsf{r}_X^{\mathscr{M}}(Y) \oplus \mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) \arrow[r] & \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}) \\
\mathsf{r}_X^{\mathscr{M}}(T) \oplus \mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(T)) \arrow[r] \arrow[u] \arrow[d] & \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,T}) \arrow[u] \arrow[d] \\
\mathsf{ra}_X^{\mathscr{M}}(T) \oplus \mathsf{T}_X^{\mathscr{M}}(\mathsf{ra}_X^{\mathscr{M}}(T)) \arrow[r] & \mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,T}).
\end{tikzcd}$$

The vertical morphisms are weak equivalences by Proposition 6.17 and Remark 6.5 and so the result follows from Lemma 6.22 which ensures that the lower horizontal morphism is a weak equivalence.

Let $1_Y : Y \to \mathbf{G}_{m,Y}$ be the unit section and $p : \mathbf{G}_{m,Y} \to Y$ be the projection. Since $p \circ 1_Y = \mathrm{Id}_Y$, the morphisms induced by the unit section

$$\mathsf{RQ}_X^{\mathscr{M}}(Y \otimes \mathbb{Q}) \to \mathsf{RQ}_X^{\mathscr{M}}(\mathbf{G}_{m,Y} \otimes \mathbb{Q}) \qquad \mathsf{r}_X^{\mathscr{M}}(Y) \to \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})$$

are monomorphisms. . We have then a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{r}_X^{\mathscr{M}}(Y) \arrow[r] & \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}) \arrow[r] & \mathsf{c}_X^{\mathscr{M}}(Y) \arrow[r] & 0 \\
0 \arrow[r] & \mathsf{r}_X^{\mathscr{M}}(Y) \arrow[r] \arrow[u] & \mathsf{r}_X^{\mathscr{M}}(Y) \oplus \mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) \arrow[r] \arrow[u] & \mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) \arrow[r] \arrow[u, dotted] & 0
\end{tikzcd}$$

in which all the rows are exact sequences. This provides the desired weak equivalence. $\square$

A
Brown-Gersten property in the Nisnevich topology

A.1. Recall that an elementary Nisnevich square, is a cartesian square in Sm/X

$$\begin{tikzcd}
V \arrow[r, "v"] \arrow[d, "e'"'] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\
U \arrow[r, "u"] & Y.
\end{tikzcd} \tag{37}$$

such that u is an open immersion and e is an étale morphism that induces an isomorphism $p^{-1}(Z) \to Z$ for the reduced scheme structures where $Z = Y \setminus U$. If e is also an open immersion then the square is called an elementary Zariski square (an elementary Zariski square is simply the data of a covering of X by two open subschemes U and

E). If all the schemes in (37) are affine then the square is called an elementary affine Nisnevich square.

If $Y \in \mathsf{Sm}/X$ is connected, a morphism of quasi-projective X-schemes $r : Y' \to Y$ is said to be a Galois cover if r is finite étale and $G := \mathrm{Aut}_Y(Y')$ operates transitively and faithfully on the geometric fibers of f. If Y is not connected then $r : Y' \to Y$ is said to be a Galois cover if its restriction to the connected components are Galois covers.

A.2. Recall some definitions from [**32, 31**]

Definition A.1. — Let $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ be a presheaf.

1. One says that $\mathscr{X}$ satisfies the B.G. property in the Zariski topology if for every $X \in \mathsf{Sm}/k$ and every covering of X by two open subschemes U, E the following diagram is homotopy cartesian in $\mathsf{Ch}(\mathbb{Q})$

$$\begin{array}{ccc} \mathscr{X}(Y) & \longrightarrow & \mathscr{X}(E) \\ \downarrow & & \downarrow \\ \mathscr{X}(U) & \longrightarrow & \mathscr{X}(V) \end{array}$$

   One says that $\mathscr{X}$ satisfies the $\mathbf{A}^1$-B.G. property in the Zarisky topology if $\mathscr{X}$ satisfies the B.G. property in the Zariski topology and for every $X \in \mathsf{Sm}/k$ the map

$$\mathscr{X}(X) \to \mathscr{X}(X \times_k \mathbf{A}^1_k),$$

   induced by the projection, is a quasi-isomorphism.

2. One says that $\mathscr{X}$ satisfies the B.G. property (resp. affine B.G. property) in the Nisnevich topology if, for every $X \in \mathsf{Sm}/k$ and every elementary Nisnevich square (resp. elementary affine Nisnevich square) (37), the following diagram is homotopy cartesian in $\mathsf{Ch}(\mathbb{Q})$

$$\begin{array}{ccc} \mathscr{X}(Y) & \longrightarrow & \mathscr{X}(E) \\ \downarrow & & \downarrow \\ \mathscr{X}(U) & \longrightarrow & \mathscr{X}(V) \end{array}$$

By [**31**, Theorem A.14], if an object $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ satisfies the $\mathbf{A}^1$-B.G. property in the Zariski topology and the affine B.G. property in the Nisnevich topology, then it satisfies the B.G. property in the Nisnevich topology.

A.3. Let $\mathscr{A}$ be a pseudo-Abelian $\mathbb{Q}$-linear additive category. Given a finite group G and an object A of $\mathscr{A}$, an action of G on A is a morphism of groups

$$\Phi_A : G \to \mathrm{Aut}_{\mathscr{A}}(A)$$

where $\mathrm{Aut}_{\mathscr{A}}(A)$ is the group of automorphism of A. Since $\mathscr{A}$ is $\mathbb{Q}$-linear, we may consider the projector

$$\Pi_G := \frac{1}{|G|}\sum_{g\in G}\Phi_A(g)$$

for any object of $\mathscr{A}$ with an action of G by automorphisms. The category $\mathscr{A}$ being pseudo-Abelian, Π_G splits providing a decomposition of A. The invariant A^G under G is the direct summand of A image of Π_G.

Definition A.2. — A presheaf $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ has elementary Galois descent if, for every Galois cover $Y' \to Y$, the morphism

$$\mathscr{X}(Y) \to \mathscr{X}(Y')^G \tag{38}$$

is a quasi-isomorphism of $\mathbb{Q}$-vector spaces.

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